

Geometric Objects - Spaces and Matrix

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Spaces

□ Vector space

- The vector space has scalars and vectors.
- Scalars: α , β , δ
- Vectors: u , v , w

□ Affine space

- The affine space has point in addition to the vector space.
- Points: P , Q , R

□ Euclidean space

- In Euclidean space, the concept of distance is added.

Scalars, Points, Vectors

- 3 basic types needed to describe the geometric objects and their relations
- Scalars: α, β, δ
- Points: P, Q, R
- Vectors: u, v, w
- Vector space
 - scalars & vectors
- Affine space
 - Extension of the vector space that includes a point

Scalars

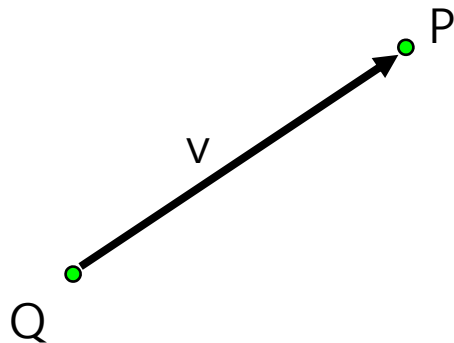
- Commutative, associative, and distribution laws are established for addition and multiplication
 - $\alpha + \beta = \beta + \alpha$
 - $\alpha \cdot \beta = \beta \cdot \alpha$
 - $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
 - $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
 - $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$
- Addition identity is 0 and multiplication identity is 1.
 - $\alpha + 0 = 0 + \alpha = \alpha$
 - $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$
- Inverse of addition and inverse of multiplication
 - $\alpha + (-\alpha) = 0$
 - $\alpha \cdot \alpha^{-1} = 1$

Vectors

- ▣ Vectors have **magnitude (or length)** and **direction**.
- ▣ Physical quantities, such as velocity or force, are vectors.
- ▣ Directed line segments used in computer graphics are vectors.
- ▣ **Vectors do not have a fixed position in space.**

Points

- Points have a position in space.
- Operations with points and vectors:
 - Point-point subtraction creates a vector.
 - Point-vector addition creates points.

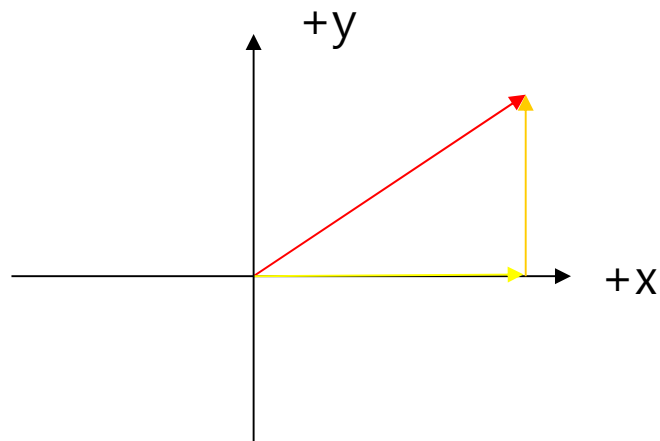


$$v = P - Q$$

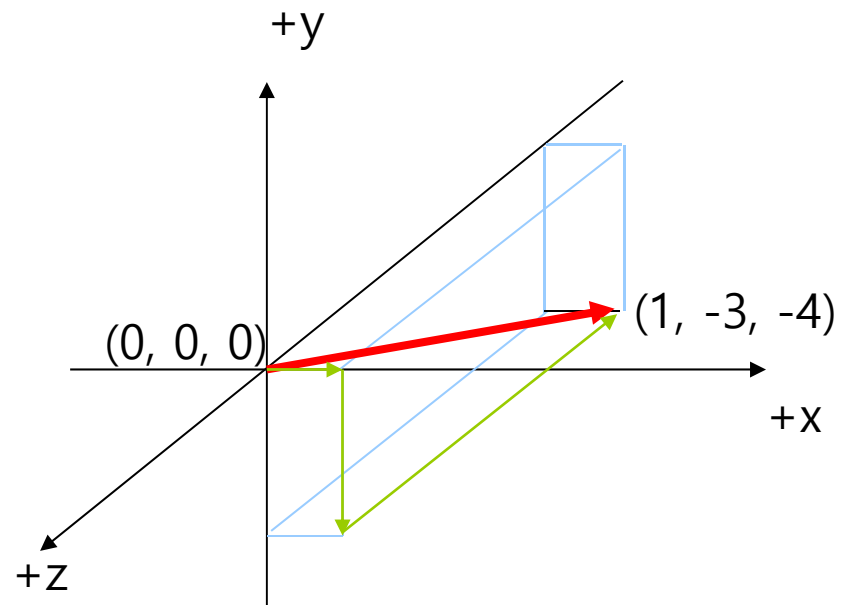
$$P = Q + v$$

Specifying Vectors

- 2D Vector: (x, y)
- 3D Vector: (x, y, z)



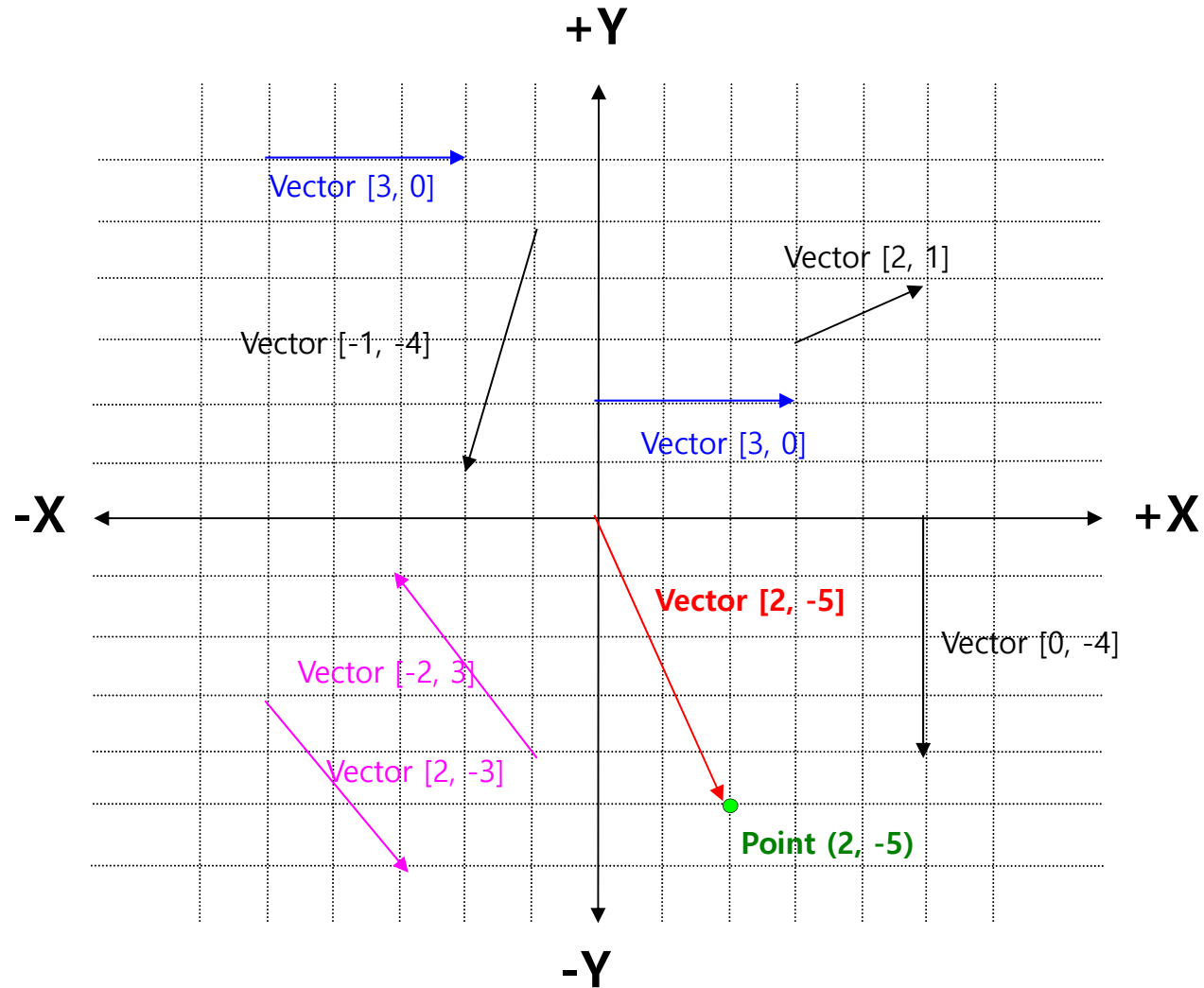
2D Vector



3D Vector

Vector from the origin $O(0, 0, 0)$
to the point $P(1, -3, -4)$

Examples of 2D vectors

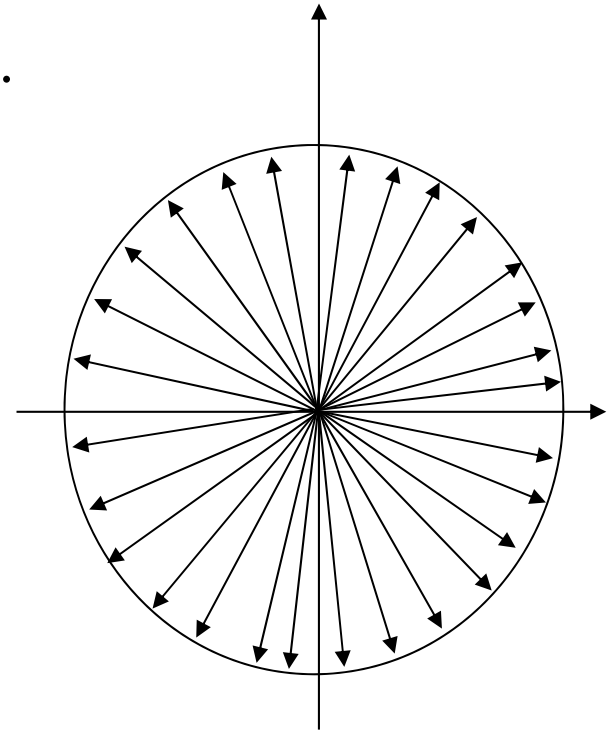


Vector Operations

- ❑ zero vector
- ❑ vector negation
- ❑ vector/scalar multiply
- ❑ add & subtract two vectors
- ❑ vector magnitude (length)
- ❑ normalized vector
- ❑ distance formula
- ❑ vector product
 - dot product
 - cross product

The Zero Vector

- The three-dimensional zero vector is $(0, 0, 0)$.
- The zero vector has **zero magnitude**.
- The zero vector has **no direction**.



Negating a Vector

□ Every vector $\mathbf{v}$ has a negative vector $-\mathbf{v}$: $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

□ Negative vector

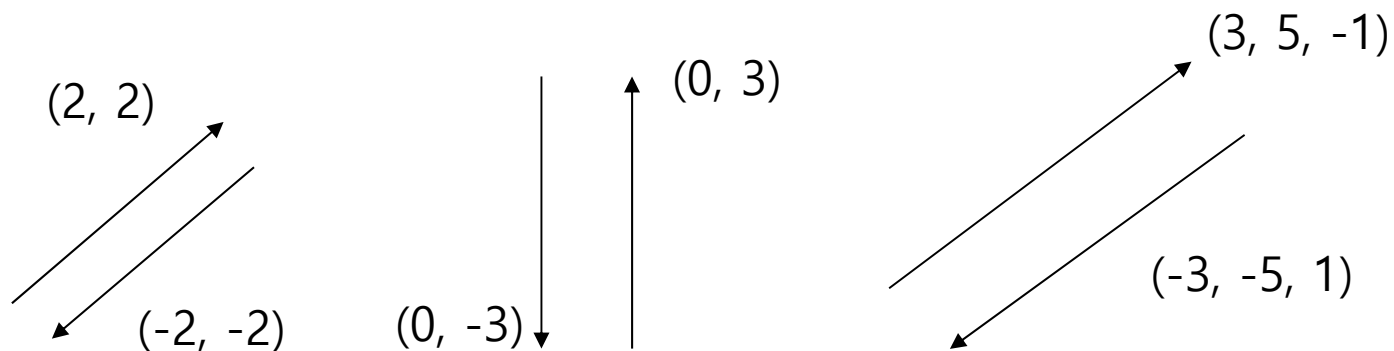
$$-(a_1, a_2, a_3, \dots, a_n) = (-a_1, -a_2, -a_3, \dots, -a_n)$$

□ 2D, 3D, 4D vector negation

$$-(x, y) = (-x, -y)$$

$$-(x, y, z) = (-x, -y, -z)$$

$$-(x, y, z, w) = (-x, -y, -z, -w)$$



Vector-Scalar Multiplication

- Vector scalar multiplication

$$\alpha * (x, y, z) = (\alpha x, \alpha y, \alpha z)$$

- Vector scale division

$$1/\alpha * (x, y, z) = (x/\alpha, y/\alpha, z/\alpha)$$

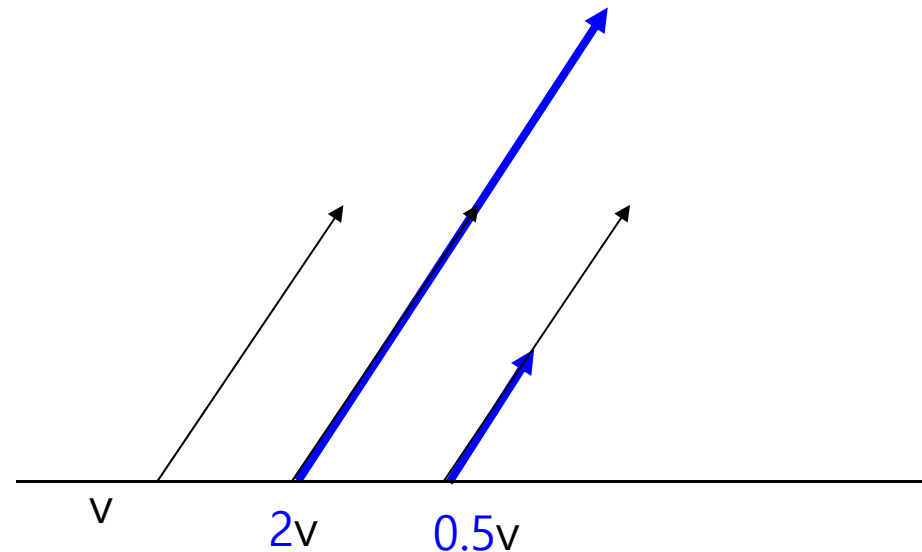
- Example:

$$2 * (4, 5, 6) = (8, 10, 12)$$

$$\frac{1}{2} * (4, 5, 6) = (2, 2.5, 3)$$

$$-3 * (-5, 0, 0.4) = (15, 0, -1.2)$$

$$3\mathbf{u} + \mathbf{v} = (3\mathbf{u}) + \mathbf{v}$$



Vector Addition and Subtraction

□ Vector Addition

- Defined as a head-to-tail axiom

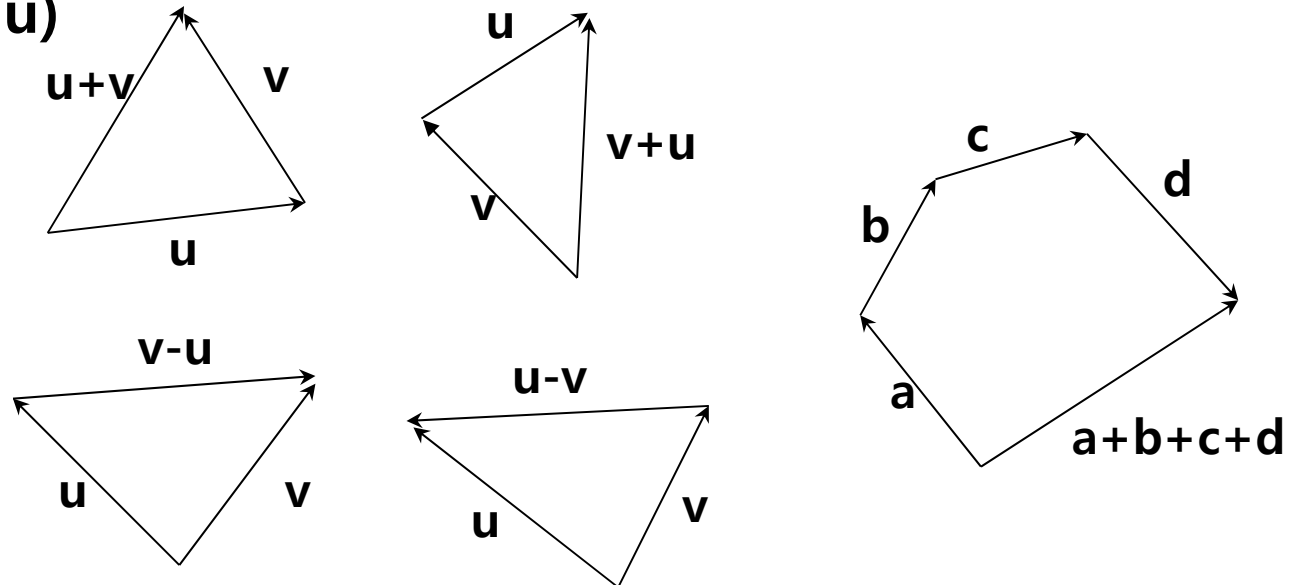
$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$$

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

□ Vector Subtraction

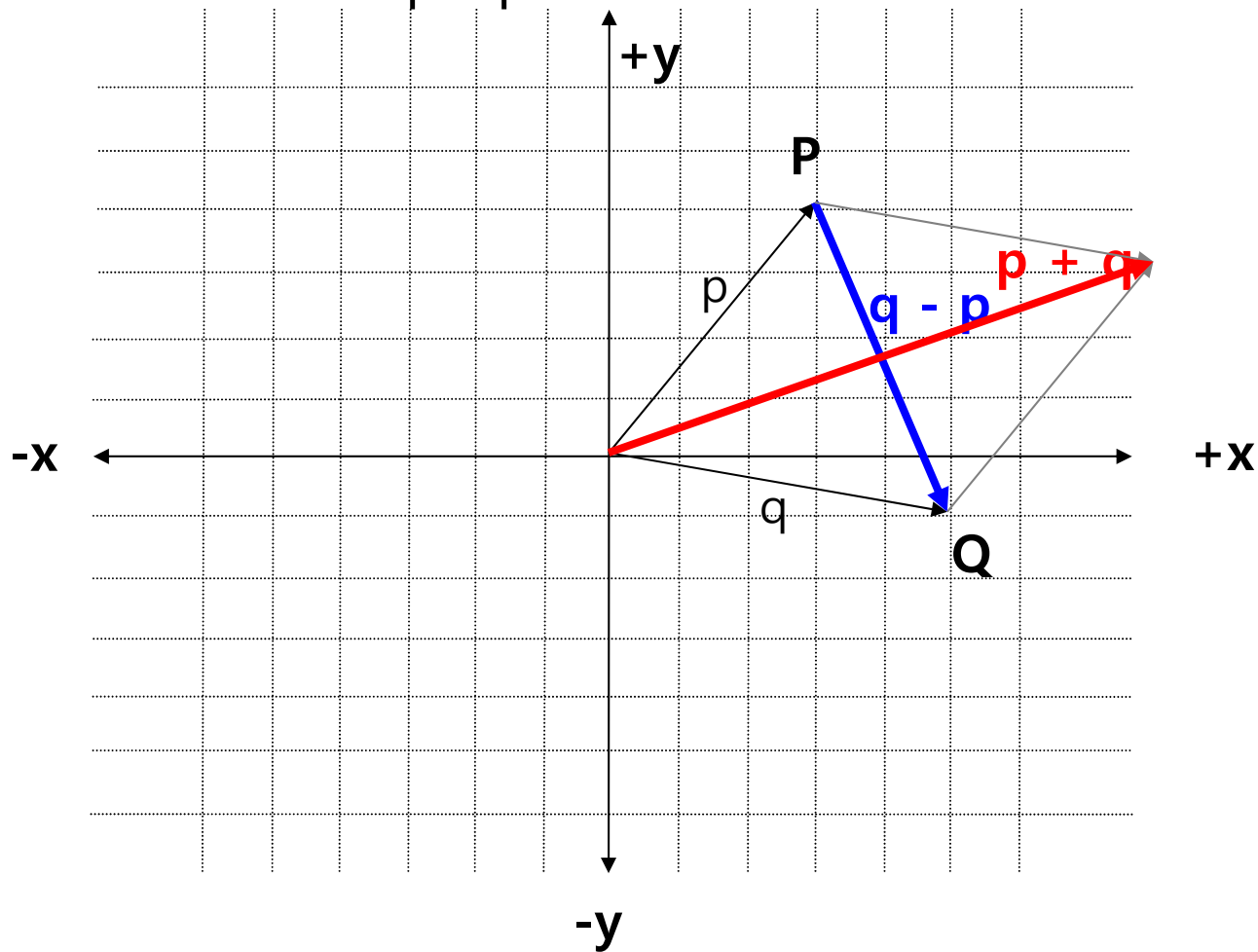
$$(x_1, y_1, z_1) - (x_2, y_2, z_2) = (x_1-x_2, y_1-y_2, z_1-z_2)$$

$$\mathbf{u} - \mathbf{v} = -(\mathbf{v} - \mathbf{u})$$



Vector Addition and Subtraction

- ▣ The displacement vector from the point P to the point Q is calculated as $q - p$.



Vector Magnitude (Length)

□ Vector magnitude (or length):

Examples: $\|v\| = \sqrt{v_1^2 + v_2^2 + \dots + v_{n-1}^2 + v_n^2}$

$$\|(5, -4, 7)\| = \sqrt{5^2 + (-4)^2 + 7^2}$$

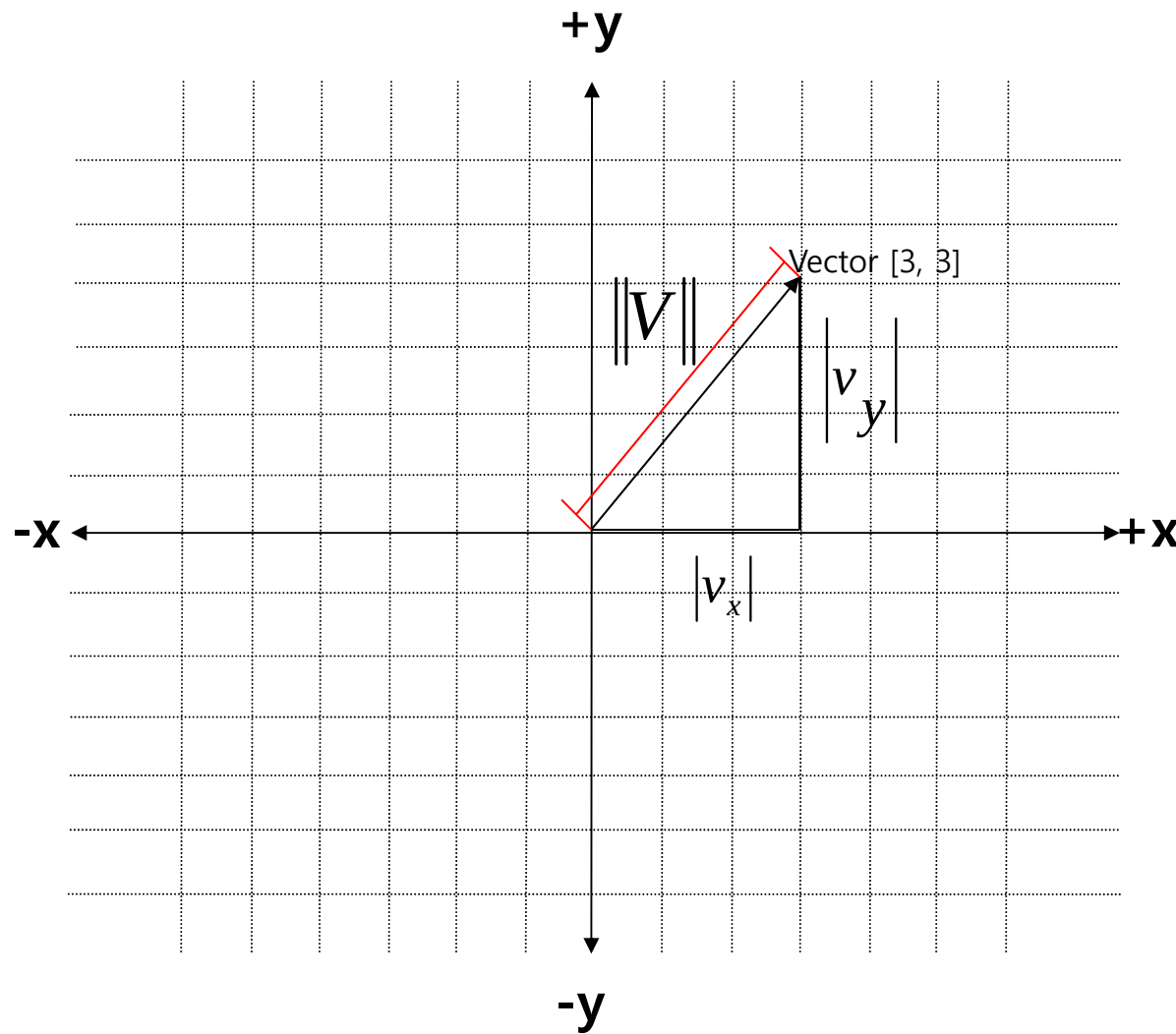
$$= \sqrt{25 + 16 + 49}$$

$$= \sqrt{90}$$

$$= 3\sqrt{10}$$

$$\approx 9.4868$$

Vector Magnitude



$$\|v\|^2 = |v_x|^2 + |v_y|^2$$

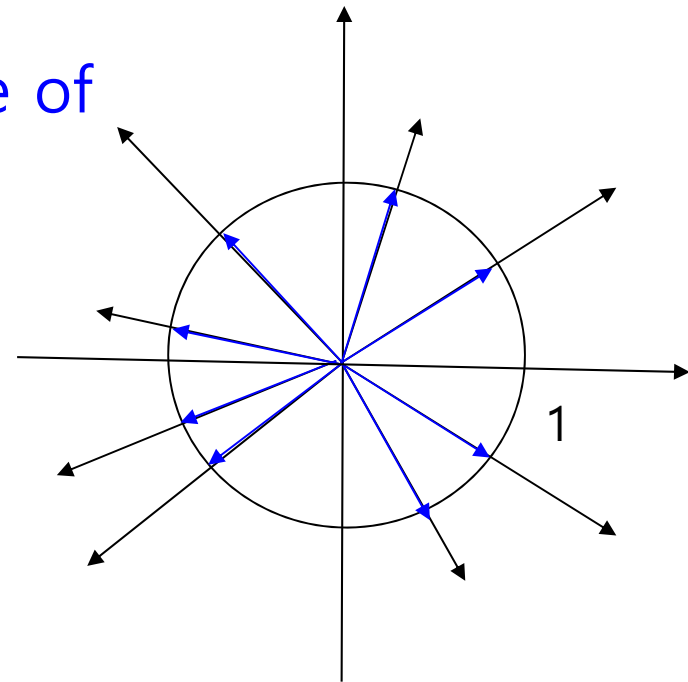
$$\sqrt{\|v\|^2} = \sqrt{v_x^2 + v_y^2}$$

$$\|v\| = \sqrt{v_x^2 + v_y^2}$$

Normalized Vectors

- ❑ There is case where you only need the direction of the vector, regardless of the vector length.
- ❑ The unit vector has a magnitude of 1.
- ❑ The unit vector is also called as *normalized vectors or normal*.
- ❑ "Normalizing" a vector:

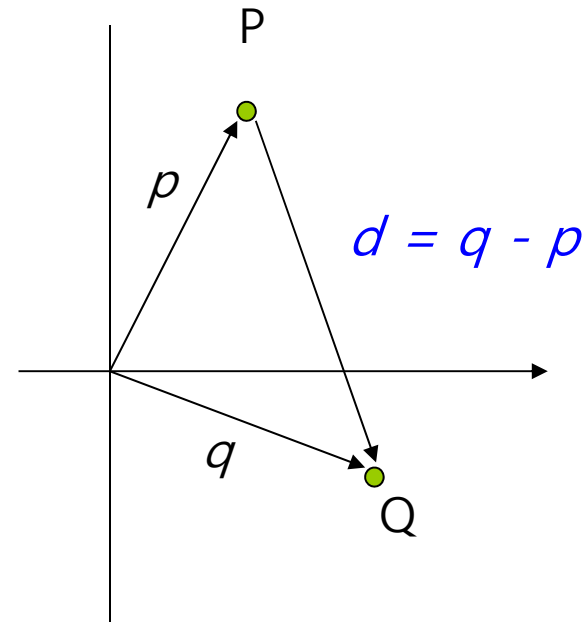
$$\mathbf{v}_{norm} = \frac{\mathbf{v}}{\|\mathbf{v}\|}, \mathbf{v} \neq 0$$



Distance

□ The distance between two points P and Q is calculated as follows.

- Vector p
- Vector q
- Displacement vector $d = q - p$
- Find the length of the vector d .
- $\text{distance}(P, Q) = \|d\| = \|q - p\|$



Vector Dot Product

- Dot product between two vectors: $\mathbf{u} \cdot \mathbf{v}$

$$(u_1, u_2, u_3, \dots, u_n) \cdot (v_1, v_2, v_3, \dots, v_n) = \\ u_1v_1 + u_2v_2 + \dots + u_{n-1}v_{n-1} + u_nv_n$$

or

$$u \cdot v = \sum_{i=1}^n u_i v_i$$

$$u \cdot u = \|u\|^2$$

- Example:

$$(4, 6) \cdot (-3, 7) = 4 \cdot (-3) + 6 \cdot 7 = 30$$

$$(3, -2, 7) \cdot (0, 4, -1) = 3 \cdot 0 + (-2) \cdot 4 + 7 \cdot (-1) = -15$$

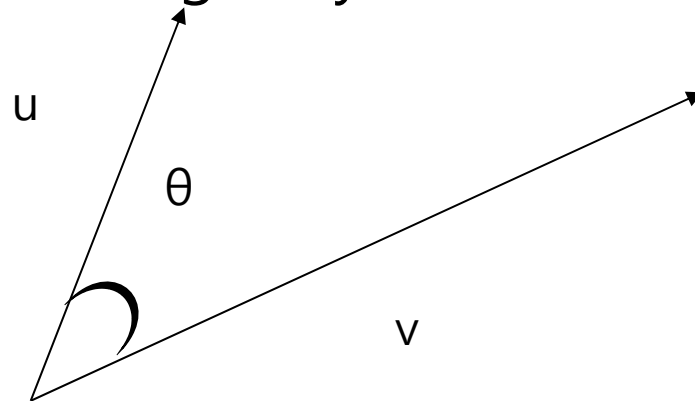
Vector Dot Product

- ▣ The dot product of the two vectors is the cosine of the angle between two vectors (assuming they are normalized).

$$u \cdot v = \|u\| \|v\| \cos \theta$$

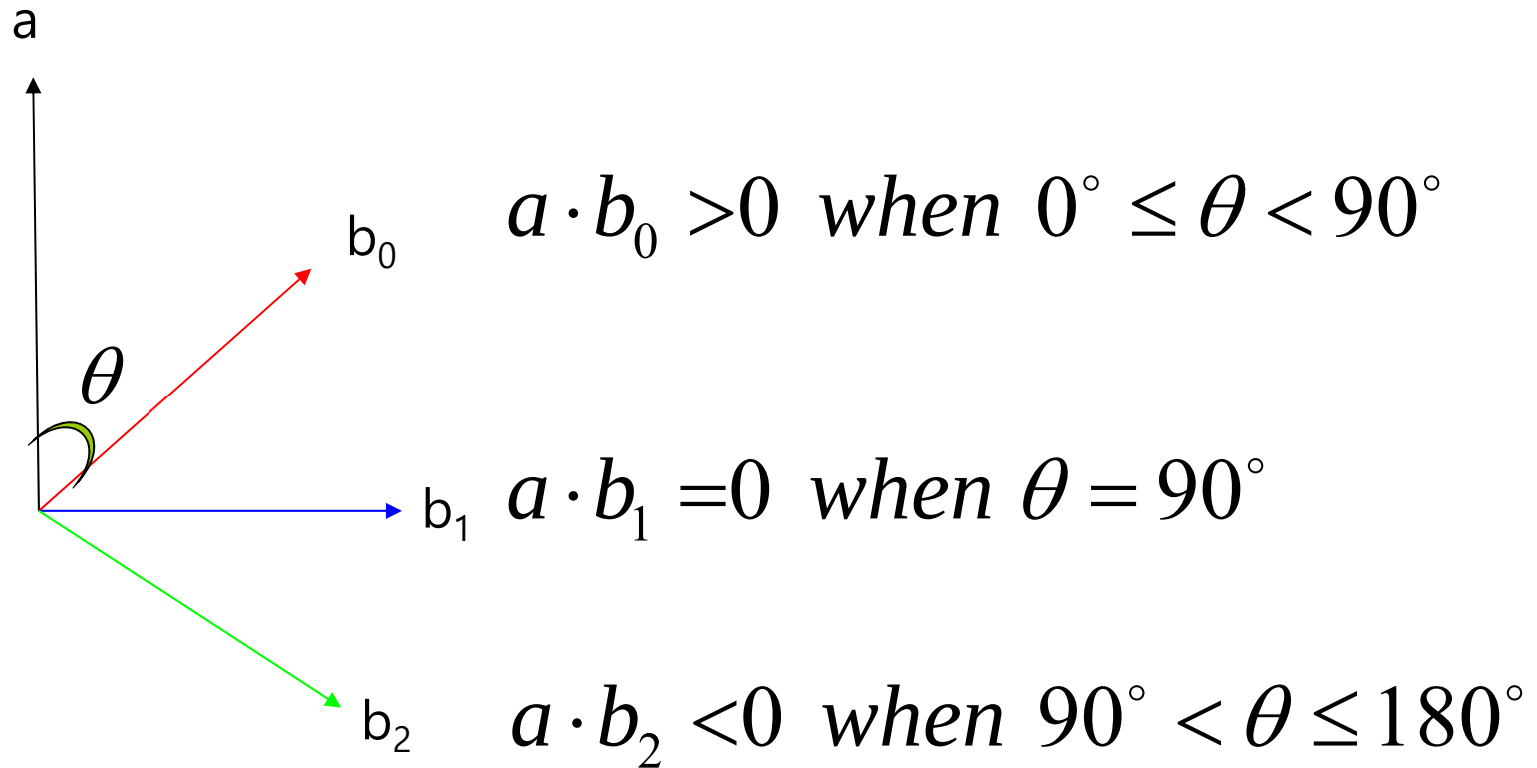
$$\theta = \arccos\left(\frac{u \cdot v}{\|u\| \|v\|}\right)$$

$$\theta = \arccos(u \cdot v), \text{ where } u, v \text{ are unit vectors}$$



Dot Product as Measurement of Angle

- ▣ The following is the characteristics of the dot product.



Projecting One Vector onto Another

- Given two vectors, w and v , one vector w can be divided into parallel and orthogonal to the other vector v .

$$W = W_{\text{par}} + W_{\text{per}}$$

$$W = \alpha V + u$$

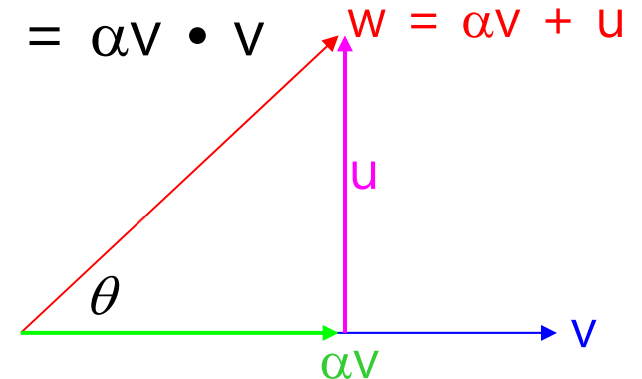
u must be orthogonal to v , $u \cdot v = 0$

$$w \cdot v = (\alpha v + u) \cdot v = \alpha v \cdot v + u \cdot v = \alpha v \cdot v$$

$$\alpha = \frac{w \cdot v}{v \cdot v}$$

$$u = w - \alpha v = w - \frac{w \cdot v}{v \cdot v} v = w - \frac{w \cdot v}{\|v\|^2} v$$

$$\alpha v = w - u = w - w + \frac{w \cdot v}{v \cdot v} v = \frac{w \cdot v}{v \cdot v} v = \frac{w \cdot v}{\|v\|^2} v$$

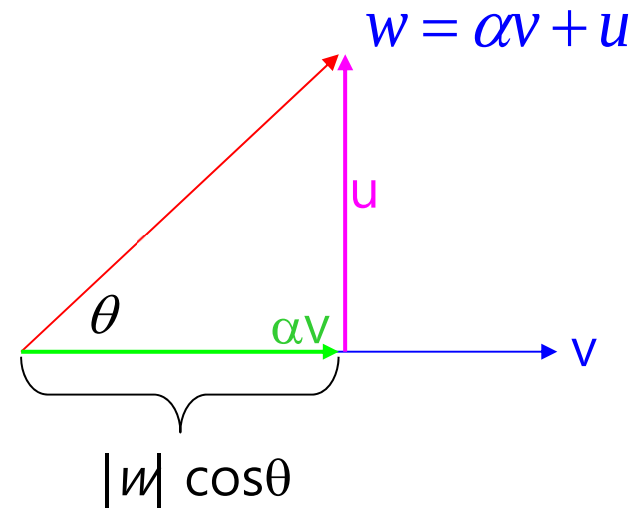


Projecting One Vector onto Another

If v is a unit vector,
then $\|v\| = 1$

$$w_{per} = u = w - (w \cdot v)v$$

$$w_{par} = \alpha v = (w \cdot v)v$$



$$\cos \theta = \frac{\|\alpha v\|}{\|w\|} \Rightarrow \|\alpha v\| = \|w\| \cos \theta$$
$$\sin \theta = \frac{\|u\|}{\|w\|} \Rightarrow \|u\| = \|w\| \sin \theta$$

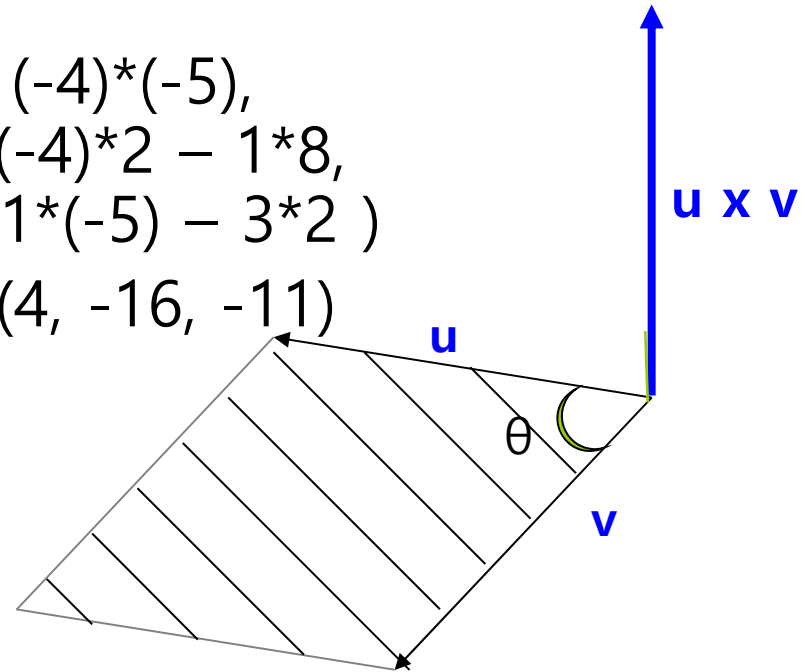
Vector Cross Product

▣ Cross product: $\mathbf{u} \times \mathbf{v}$

$$\begin{aligned} (x_1, y_1, z_1) \times (x_2, y_2, z_2) = & (y_1z_2 - z_1y_2, \\ & z_1x_2 - x_1z_2, \\ & x_1y_2 - y_1x_2) \end{aligned}$$

▣ Example:

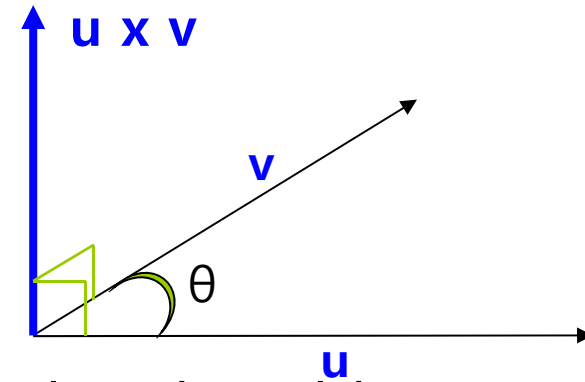
$$\begin{aligned} (1, 3, -4) \times (2, -5, 8) = & (3 \cdot 8 - (-4) \cdot (-5), \\ & (-4) \cdot 2 - 1 \cdot 8, \\ & 1 \cdot (-5) - 3 \cdot 2) \\ = & (4, -16, -11) \end{aligned}$$



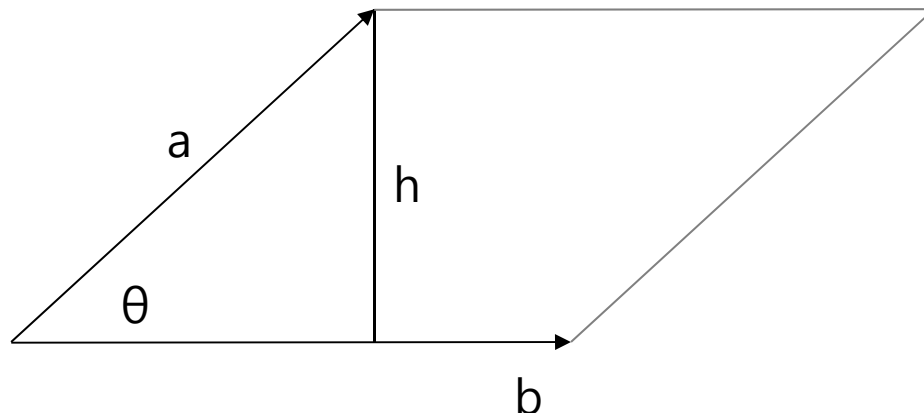
Vector Cross Product

- The magnitude of the cross product between two vectors, $|\mathbf{u} \times \mathbf{v}|$, is the product of the magnitude of each other and the sine of the angle between the two vectors.

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$



- The area of the parallelogram is calculated as bh .



$$A = bh$$

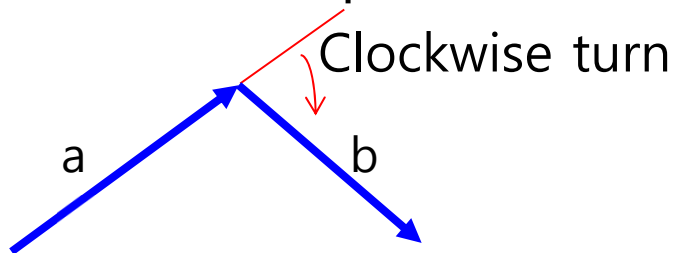
$$= b (a \sin \theta)$$

$$= \|a\| \|b\| \sin \theta$$

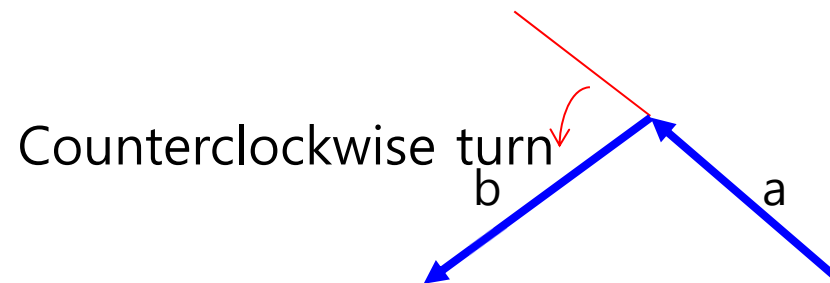
$$= \|\mathbf{a} \times \mathbf{b}\|$$

Vector Cross Product

- In the left-handed coordinate system, when the vectors u and v move in a clockwise turn, $u \times v$ points in the direction toward us, and when moving in a counter-clockwise turn, $u \times v$ points in the direction away from us.
- In the right-handed coordinate system, when the vectors u and v move in a counter-clockwise turn, $u \times v$ points in the direction toward us, and when moving in a clockwise turn, $u \times v$ points in the direction away from us.



Left-handed Coordinates



Right-handed Coordinates

Linear Algebra Identities

Identity	Comments
$u + v = v + u$	Vector addition commutative law
$u - v = u + (-v)$	Vector subtraction
$(u+v)+w = u+(v+w)$	Vector addition associative law
$\alpha(\beta u) = (\alpha\beta)u$	Scalar-Vector multiplication association
$\alpha(u + v) = \alpha u + \alpha v$ $(\alpha + \beta)u = \alpha u + \beta u$	Scalar-Vector distribution law
$\ \alpha v\ = \alpha \ v\ $	Scalar product
$\ v\ \geq 0$	The magnitude of vector is nonnegative
$\ u\ ^2 + \ v\ ^2 = \ u + v\ ^2$	Pythagorean theorem
$\ u\ + \ v\ \geq \ u + v\ $	Vector addition triangle rule
$u \cdot v = v \cdot u$	Dot product commutative law
$\ v\ = \sqrt{v \cdot v}$	Vector magnitude using dot product

Linear Algebra Identities

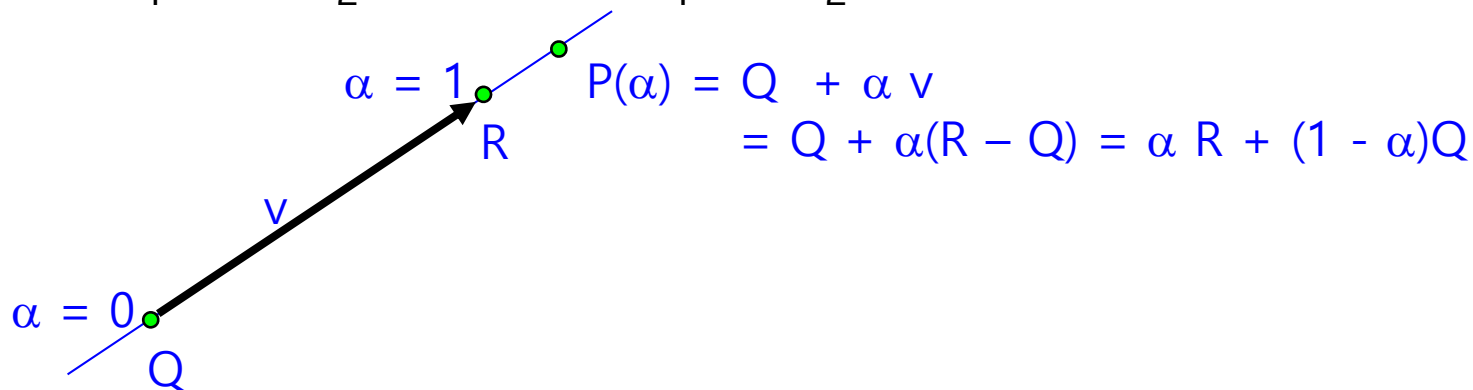
Identity	Comments
$\alpha(u \cdot v) = (\alpha u) \cdot v = u \cdot (\alpha v)$	Vector dot product and scalar product associative law
$u \cdot (v + w) = u \cdot v + u \cdot w$	Vector addition and dot product distribution law
$u \times u = 0$	Cross product of the vector itself is 0.
$u \times v = -(v \times u)$	Cross product is anti-commutative.
$u \times v = (-u) \times (-v)$	Cross product of a vector is equal to the cross product of inverse of each vector.
$\alpha(u \times v) = (\alpha u) \times v = u \times (\alpha v)$	Scalar and cross product multiplication associative law
$u \times (v+w) = (u \times v) + (u \times w)$	Cross product of vector and the addition of two vector establish the distribution law
$u \cdot (u \times v) = 0$	Dot product of any vector with cross product of that vector & another vector is 0

Geometric Objects

- Line
 - 2 points
- Plane
 - 3 points
- 3D objects
 - Defined by a set of triangles
 - Simple convex flat polygons
 - hollow

Lines

- Line is point-vector addition (or subtraction of two points).
- Line parametric form: $P(\alpha) = P_0 + \alpha v$
 - P_0 is arbitrary point, and v is arbitrary vector
 - Points are created on a straight line by changing the parameter.
- $v = R - Q$
$$P = Q + \alpha v = Q + \alpha(R - Q) = \alpha R + (1 - \alpha)Q$$
- $P = \alpha_1 R + \alpha_2 Q$ where $\alpha_1 + \alpha_2 = 1$



Lines, Rays, Line Segments

- The line is infinitely long in both directions.
- A line segment is a piece of line between two endpoints. $0 \leq \alpha \leq 1$
- A ray has one end point and continues infinitely in one direction. $\alpha \geq 0$

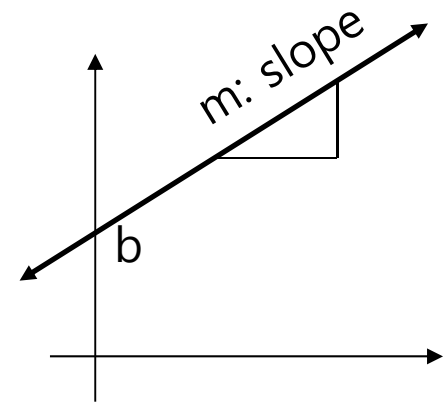
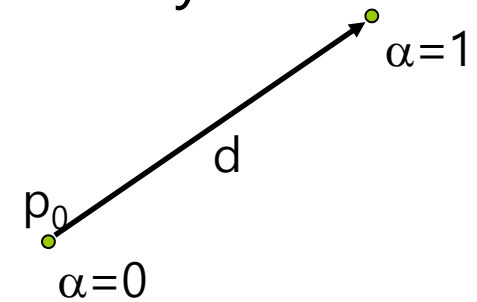
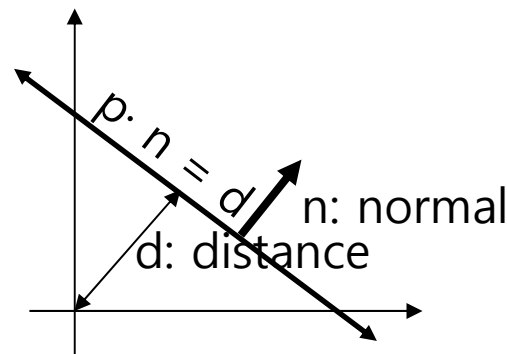
- Line:

$$p(\alpha) = p_0 + \alpha d \text{ (parametric)}$$

$$y = mx + b \text{ (explicit)}$$

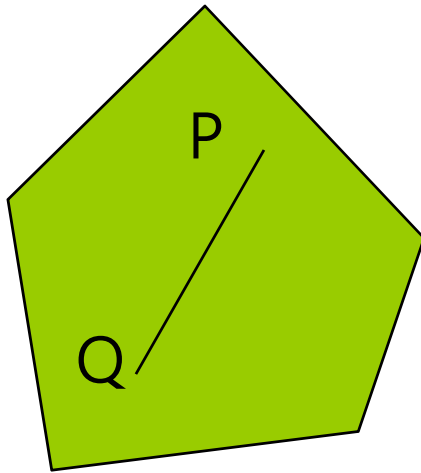
$$ax + by = d \text{ (implicit)}$$

$$p \cdot n = d$$

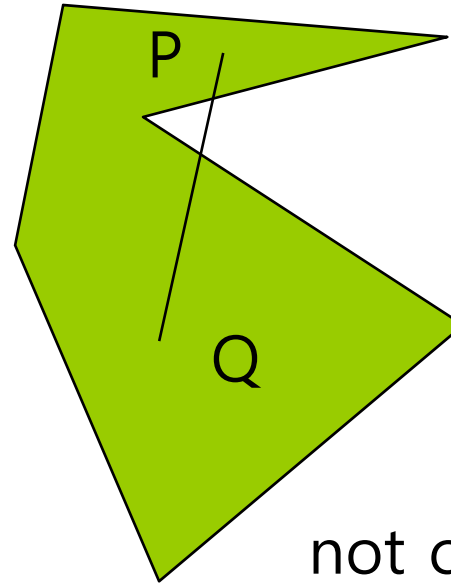


Convexity

- An object is *convex* if only if for any two points in the object all points on the line segment between these points are also in the object.



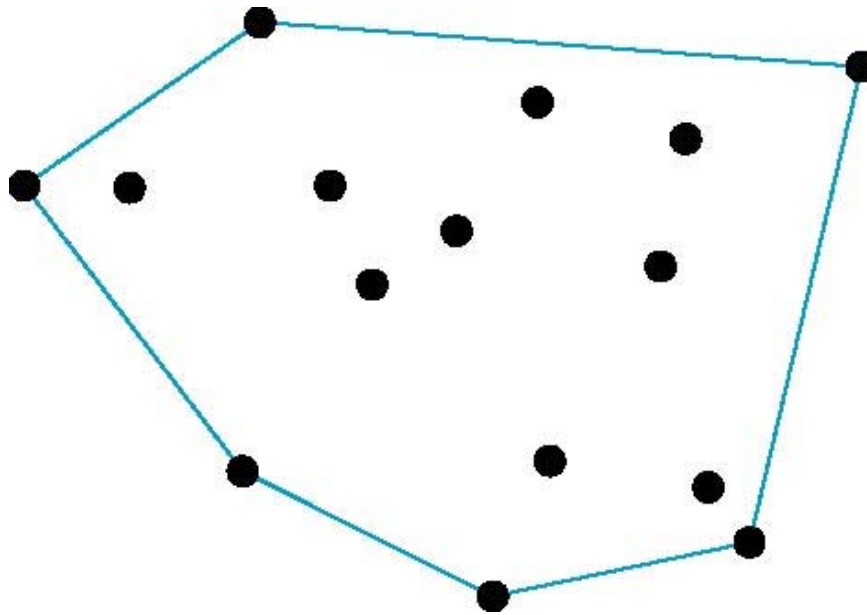
convex



not convex

Convex Hull

- **Smallest convex object** containing $P_1, P_2, \dots, P_n$
- Formed by “**shrink wrapping**” points



Affine Sums

- The affine sum of the points defined by $P_1, P_2, \dots, P_n$ is

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_n P_n$$

Can show by induction that this sum makes sense iff

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$$

- If, in addition, $\alpha_i \geq 0$, $i=1, 2, \dots, n$, we have the **convex hull** of $P_1, P_2, \dots, P_n$.
- Convex hull $\{P_1, P_2, \dots, P_n\}$, you can see that it includes all the line segments connecting the pairs of points.

Linear/Affine Combination of Vectors

- Linear combination of m vectors
 - Vector $v_1, v_2, \dots, v_m$
 - $w = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m$ where $\alpha_1, \alpha_2, \dots, \alpha_m$ are scalars
- If the sum of the scalar values, $\alpha_1, \alpha_2, \dots, \alpha_m$ is 1, it becomes an affine combination.
 - $\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$

Convex Combination

- If, in addition, $\alpha_i \geq 0$, $i=1,2, \dots, n$, we have the **convex hull** of $P_1, P_2, \dots, P_n$.
- Therefore, the linear combination of vectors satisfying the following condition is a convex.
 - $\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$
 - and
 - $\alpha_i \geq 0$ for $i=1,2, \dots, m$
 - α_i is between 0 and 1
- Convexity
 - Convex hull

Plane

- A plane can be defined by a point and two vectors or by three points.

- Suppose 3 points, P, Q, R

- Line segment PQ

- $S(\alpha) = \alpha P + (1 - \alpha)Q$

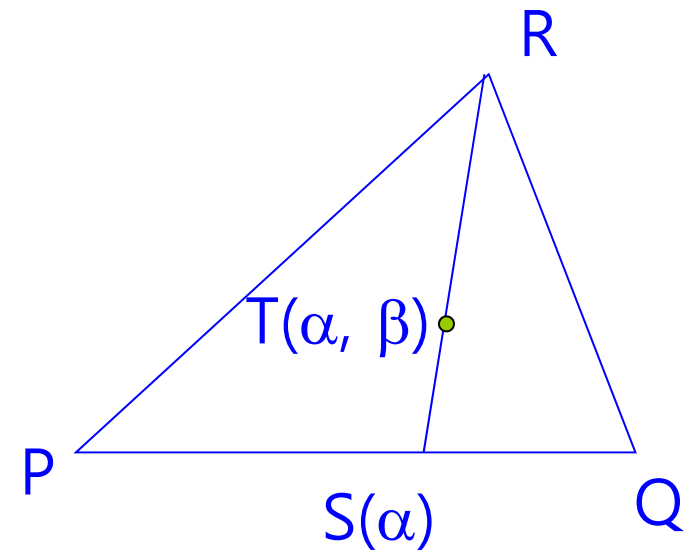
- Line segment SR

- $T(\beta) = \beta S + (1 - \beta)R$

- Plane defined by P, Q, R

- $$\begin{aligned} T(\alpha, \beta) &= \beta(\alpha P + (1 - \alpha)Q) + (1 - \beta)R \\ &= P + \beta(1 - \alpha)(Q - P) + (1 - \beta)(R - P) \end{aligned}$$

- For $0 \leq \alpha, \beta \leq 1$, we get all points in triangle, $T(\alpha, \beta)$.



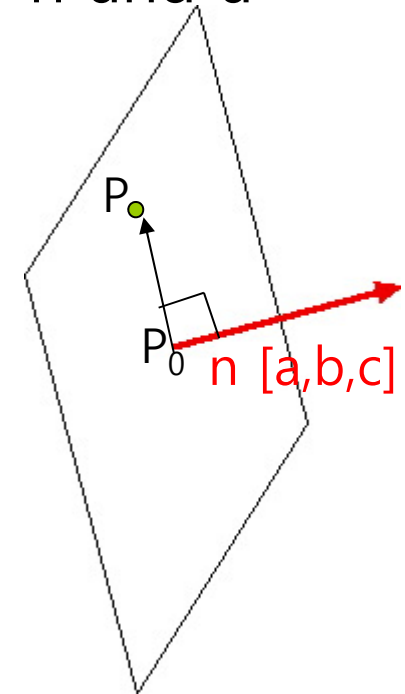
Plane

- Plane equation defined by a point P_0 and two non parallel vectors, u , v
 - $T(\alpha, \beta) = P_0 + \alpha u + \beta v$
 - $P - P_0 = \alpha u + \beta v$ (P is a point on the plane)
- Using n (the cross product of u , v), the plane equation is as follows
 - $n \cdot (P - P_0) = 0$ (where $n = u \times v$ and n is a normal vector)

Plane

- The plane is represented by a normal vector n and a point P_0 on the plane.

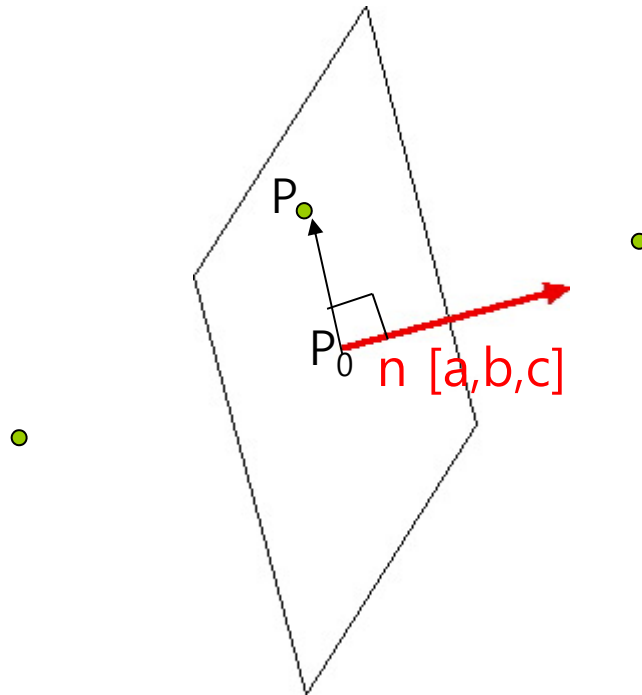
- Plane (n, d) where $n (a, b, c)$
- $ax + by + cz + d = 0$
- $n \cdot p + d = 0$
 $d = -n \cdot p$



- For point p on the plane, $n \cdot (p - p_0) = 0$
- If the plane normal n is a unit vector, then $n \cdot p + d$ gives the shortest signed distance from the plane to point p : $d = -n \cdot p$

Relationship between Point and Plane

- Relationship between point p and plane (n, d)
 - If $n \cdot p + d = 0$, then p is in the plane.
 - If $n \cdot p + d > 0$, then p is outside the plane.
 - If $n \cdot p + d < 0$, then p is inside the plane.



Plane Normalization

□ Plane normalization

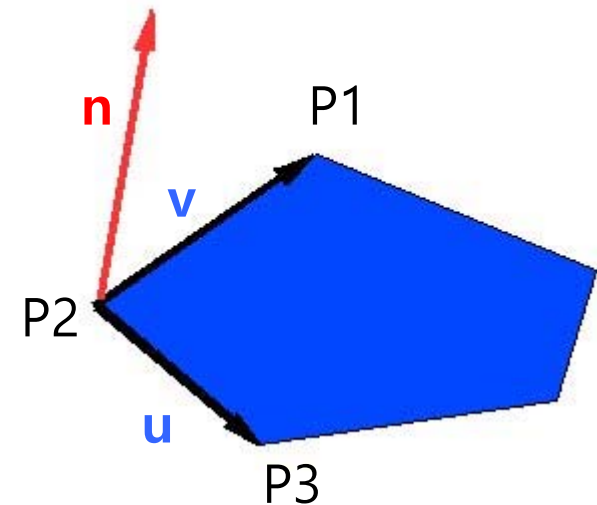
- Normalize the plane normal vector
- Since the length of the normal vector affects the constant d , d is also normalized.

$$\frac{1}{\|n\|} (n, d) = \left(\frac{n}{\|n\|}, \frac{d}{\|n\|} \right)$$

Computing a Normal from 3 Points in Plane

- Find the normal from the polygon's vertices.
 - The polygon's normal computes two non-collinear edges. (assuming that no two adjacent edges will be collinear)
 - Then, normalize it after the cross product.

```
void computeNormal(vector P1, vector P2, vector P3) {  
    vector u, v, n, y(0, 1, 0);  
    u = P3 - P2;  
    v = P1 - P2;  
    n = cross(u, v);  
    if (n.length()==0)  
        return y;  
    else  
        return n.normalize();  
}
```



Computing a Distance from Point to Plane

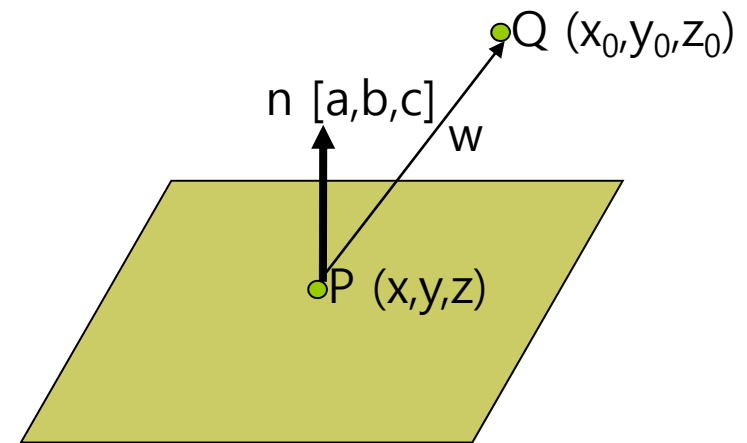
- Find the closest distance to a plane (n, d) in space and a point Q out of the plane.
 - The plane's normal is n , and D is the distance between a point P and a point Q on the plane.

$$w = Q - P = [x_0 - x, y_0 - y, z_0 - z]$$

$$D = \frac{|n \cdot w|}{\|n\|}$$

$$= \frac{|a(x_0 - x) + b(y_0 - y) + c(z_0 - z)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}$$



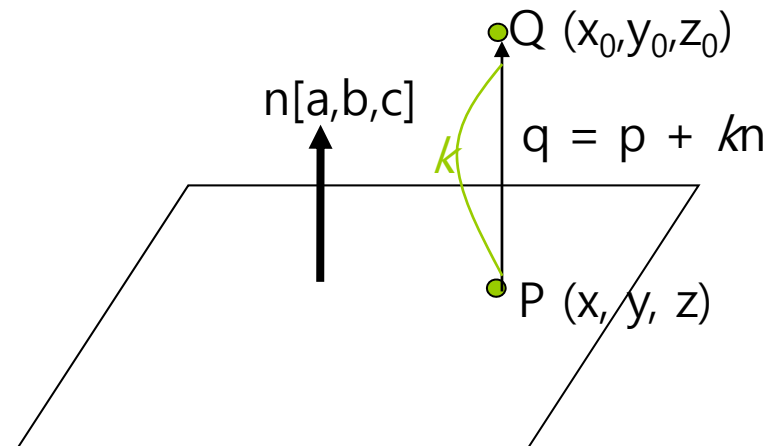
$$\text{Projecting } w \text{ onto } n : w_{\parallel} = n \frac{w \cdot n}{\|n\|^2} \ \& \ \|w_{\parallel}\| = \frac{|w \cdot n|}{\|n\|}$$

Closest Point on the Plane

- Find a point **P** on the plane (n, d) closest to one point **Q** in space.

- $p = q - kn$ (k is the shortest signed distance from point Q to the plane)

- If n is a unit vector,
 $k = n \cdot q + d$
 $p = q - (n \cdot q + d)n$



$$\text{Distance}(q, \text{plane}) = \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}$$

where $q(x_0, y_0, z_0)$ and Plane $ax + by + cz + d = 0$

$$\text{Distance}(q, \text{plane}) = n \cdot q + d \quad (n \text{ is a unit vector})$$

Intersection of Ray and Plane

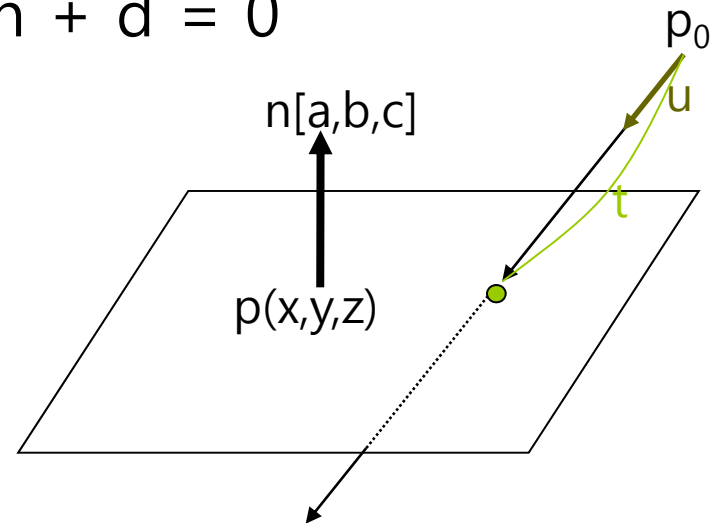
□ Ray $\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{u}$ & plane $\mathbf{p} \cdot \mathbf{n} + d = 0$

□ Ray/Plane intersection:

$$(\mathbf{p}_0 + t\mathbf{u}) \cdot \mathbf{n} + d = 0$$

$$t\mathbf{u} \cdot \mathbf{n} = -d - \mathbf{p}_0 \cdot \mathbf{n}$$

$$t = \frac{-(\mathbf{p}_0 \cdot \mathbf{n} + d)}{\mathbf{u} \cdot \mathbf{n}}$$



□ If the ray is parallel to the plane, the denominator $\mathbf{u} \cdot \mathbf{n} = 0$. Thus, the ray does not intersect the plane.

□ If the value of t is not in the range $[0, \infty)$, the ray does not intersect the plane.

□
$$\mathbf{p}\left(\frac{-(\mathbf{p}_0 \cdot \mathbf{n} + d)}{\mathbf{u} \cdot \mathbf{n}}\right) = \mathbf{p}_0 + \frac{-(\mathbf{p}_0 \cdot \mathbf{n} + d)}{\mathbf{u} \cdot \mathbf{n}} \mathbf{u}$$

Matrix

- Matrix M ($r \times c$ matrix)
 - **Row** of horizontally arranged matrix elements
 - **Column** of vertically arranged matrix elements
 - M_{ij} is the **element** in row i and column j

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{matrix} \text{ } \\ \text{ } \\ \text{ } \end{matrix} \left. \vphantom{\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}} \right\} r(2) \text{ rows}$$

$\underbrace{\hspace{10em}}_{c(2) \text{ columns}}$

Matrix

**2x5
matrix**

$$\begin{pmatrix} 2 & -4 & 7 & 7/8 & 8 \\ -3 & 4 & 3/8 & 0 & 1 \end{pmatrix}$$

$m_{12} = -4$

M_{ij} is the **element** in row i and column j

**4x3
matrix**

$$\begin{pmatrix} 4 & 0 & 12 \\ -5 & 4 & 3 \\ 12 & 3/8 & -1 \\ 1/2 & 18 & 0 \end{pmatrix}$$

$m_{42} = 18$

Square Matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{33} & m_{33} \end{pmatrix}$$

Nondiagonal elements

Diagonal elements

- The $n \times n$ matrix is called an n -th square matrix. e.g. 2×2 , 3×3 , 4×4
- Diagonal elements vs. Non-diagonal elements

Identity Matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The identity matrix is expressed as I .
- All of the diagonals are 1, the remaining elements are 0 in $n \times n$ square matrix.
- $M I = I M = M$

Vectors as Matrices

- The n-dimension vector is expressed as a $1 \times n$ matrix or an $n \times 1$ matrix.
 - $1 \times n$ matrix is a row vector (also called a row matrix)
 - $n \times 1$ matrix is a column vector (also called a column matrix)

$$\mathbf{A} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

Transpose Matrix

- ▣ Transpose of M (rxc matrix) is denoted by M^T and is converted to cxr matrix.
 - $M^T_{ij} = M_{ji}$
 - $(M^T)^T = M$
 - $D^T = D$ for any diagonal matrix D .

$$\begin{pmatrix} a & m & c \\ d & e & f \\ g & h & i \end{pmatrix}^T = \begin{pmatrix} a & d & g \\ m & e & h \\ c & f & i \end{pmatrix}$$

Transposing Matrix

$$\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^T = \begin{bmatrix} x & y & z \end{bmatrix}$$

Matrix Scalar Multiplication

- Multiplying a matrix **M** with a scalar $\alpha = \alpha \mathbf{M}$

$$\alpha \mathbf{M} = \alpha \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{33} & m_{33} \end{pmatrix} = \begin{pmatrix} \alpha m_{11} & \alpha m_{12} & \alpha m_{13} \\ \alpha m_{21} & \alpha m_{22} & \alpha m_{23} \\ \alpha m_{31} & \alpha m_{33} & \alpha m_{33} \end{pmatrix}$$

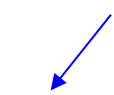
Two Matrices Addition

- Matrix C is the addition of A ($r \times c$ matrix) and B ($r \times c$ matrix), which is a $r \times c$ matrix.
- Each element c_{ij} is the sum of the ij^{th} element of A and the ij^{th} element of B.
- $c_{ij} = a_{ij} + b_{ij}$

$$\begin{pmatrix} \boxed{1} & 3 & 6 \\ 10 & 0 & -5 \\ 4 & 7 & 2 \end{pmatrix} + \begin{pmatrix} \boxed{3} & 7 & 1 \\ 6 & 4 & 9 \\ 8 & -9 & 4 \end{pmatrix} = \begin{pmatrix} \boxed{4} & 10 & 7 \\ 16 & 4 & 4 \\ 12 & -2 & 6 \end{pmatrix}$$

$r \times c \qquad \qquad \qquad r \times c \qquad \qquad \qquad r \times c$

$1+3$



Two Matrices Multiplication

- Matrix C(**rxn matrix**) is the product of A (**rxn matrix**) and B (**nxc matrix**).
- Each element c_{ij} is the vector dot product of the i^{th} row of A and the j^{th} column of B.

□
$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

$$\begin{pmatrix} 1 & 3 & 6 \\ 10 & 0 & -5 \\ 4 & 7 & 2 \end{pmatrix} * \begin{pmatrix} 3 & 7 & 1 \\ 6 & 4 & 9 \\ 8 & -9 & 4 \end{pmatrix} = \begin{pmatrix} 69 & -35 & 52 \\ -10 & 115 & -10 \\ 70 & 38 & 75 \end{pmatrix}$$

$3+18+48$

$r \times n$ $n \times c$ $r \times c$

must match *columns in result*

rows in result

Multiplying Two Matrices

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \end{pmatrix}$$

$$c_{24} = a_{21}m_{14} + a_{22}m_{24}$$

Matrix Operation

- $MI = IM = M$ (I is identity matrix)
- $A + B = B + A$: matrix addition commutative law
- $A + (B + C) = (A + B) + C$: matrix addition associative law
- $AB \neq BA$: Not hold matrix product commutative law
- $(AB)C = A(BC)$: matrix product associative law
- $ABCDEF = (((((AB)C)D)E)F) = A((((BC)D)E)F) = (AB)(CD)(EF)$
- $\alpha(AB) = (\alpha A)B = A(\alpha B)$: Scalar-matrix product
- $\alpha(\beta A) = (\alpha\beta)A$
- $(vA)B = v(AB)$
- $(AB)^T = B^T A^T$
- $(M_1 M_2 M_3 \dots M_{n-1} M_n)^T = M_n^T M_{n-1}^T \dots M_3^T M_2^T M_1^T$

Matrix Determinant

- The determinant of a square matrix M is denoted by $|M|$ or “**det M** ”.
- The determinant of non-square matrix is not defined.

$$|M| = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11} m_{22} - m_{12} m_{21}$$

$$|M| = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} = m_{11} (m_{22} m_{33} - m_{23} m_{32}) + m_{12} (m_{23} m_{31} - m_{21} m_{33}) + m_{13} (m_{21} m_{32} - m_{22} m_{31})$$

Inverse Matrix

- Inverse of M (square matrix) is denoted by M^{-1} .
- $M^{-1} = \frac{\text{adj}M}{|M|}$
- $(M^{-1})^{-1} = M$
- $M(M^{-1}) = M^{-1}M = I$
- The determinant of a non-singular matrix (i.e, invertible) is nonzero.
- The *adjoint* of M , denoted “**adj M** ” is **the transpose of the matrix of cofactors**.

$$\text{adj}M = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T$$

Cofactor of a Square Matrix & Computing Determinant using Cofactor

□ *Cofactor* of a square matrix M at a given row and column is the signed determinant of the corresponding *Minor* of M .

□ $C_{ij} = (-1)^{i+j} |M^{\{ij\}}|$

□ Calculation of $n \times n$ determinant using cofactor:

$$|M| = \sum_{j=1}^n m_{ij} c_{ij} = \sum_{j=1}^n m_{ij} (-1)^{i+j} |M^{\{ij\}}|$$

$$|M| = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} = m_{11} \begin{pmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{pmatrix} - m_{12} |M^{\{12\}}| + m_{13} |M^{\{13\}}| - m_{14} |M^{\{14\}}|$$

Minor of a Matrix

- The submatrix $M^{\{ij\}}$ is known as a minor of M , obtained by deleting row i and column j from M .

$$M = \begin{pmatrix} -4 & -3 & 3 \\ 0 & 2 & -2 \\ 1 & 4 & -1 \end{pmatrix} \quad M^{\{12\}} = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$$

Determinant, Cofactor, Inverse Matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$\det M = m_{11}m_{22} - m_{12}m_{21}$$

$$C = \begin{pmatrix} m_{22} & -m_{21} \\ -m_{12} & m_{11} \end{pmatrix}$$

$$\text{adj}M = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

$$M^{-1} = \frac{1}{\det M} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

Determinant, Cofactor, Inverse Matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

$$\det M = m_{11}(m_{22}m_{33} - m_{23}m_{32}) \\ - m_{12}(m_{21}m_{33} - m_{23}m_{31}) \\ + m_{13}(m_{21}m_{32} - m_{22}m_{31})$$

$$C = \begin{pmatrix} (m_{22}m_{33} - m_{23}m_{32}) & -(m_{21}m_{33} - m_{23}m_{31}) & (m_{21}m_{32} - m_{22}m_{31}) \\ -(m_{12}m_{33} - m_{13}m_{32}) & (m_{11}m_{33} - m_{13}m_{31}) & -(m_{11}m_{32} - m_{21}m_{31}) \\ (m_{12}m_{23} - m_{22}m_{13}) & -(m_{11}m_{23} - m_{13}m_{21}) & (m_{11}m_{22} - m_{12}m_{21}) \end{pmatrix}$$

$$\text{adj}M = \begin{pmatrix} (m_{22}m_{33} - m_{23}m_{32}) & -(m_{12}m_{33} - m_{13}m_{32}) & (m_{12}m_{23} - m_{22}m_{13}) \\ -(m_{21}m_{33} - m_{23}m_{31}) & (m_{11}m_{33} - m_{13}m_{31}) & -(m_{11}m_{23} - m_{13}m_{21}) \\ (m_{21}m_{32} - m_{22}m_{31}) & -(m_{11}m_{32} - m_{21}m_{31}) & (m_{11}m_{22} - m_{12}m_{21}) \end{pmatrix}$$

$$M^{-1} = \frac{\text{adj}M}{\det M}$$

Multiplying a Vector and a Matrix

$$\begin{aligned} \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{pmatrix} \\ = \begin{pmatrix} xp_x + yq_x + zr_x & xp_y + yq_y + zr_y & xp_z + yq_z + zr_z \end{pmatrix} \\ = \mathbf{x}\mathbf{p} + \mathbf{y}\mathbf{q} + \mathbf{z}\mathbf{r} \end{aligned}$$

- A coordinate space transformation can be expressed using a vector-matrix product.

$\mathbf{uM} = \mathbf{v}$ // matrix M converts vector u to vector v

Multiplying a Vector and a Matrix

- ▣ Vector-matrix multiplication in OpenGL (Column-Major Order)

$\mathbf{v} = \mathbf{M} * \mathbf{u}$ // matrix M converts vector u to vector v

$$\mathbf{v} = \mathbf{M} * \mathbf{u}$$

$$\begin{pmatrix} x m_{11} + y m_{12} + z m_{13} \\ x m_{21} + y m_{22} + z m_{23} \\ x m_{31} + y m_{32} + z m_{33} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$