Transformation

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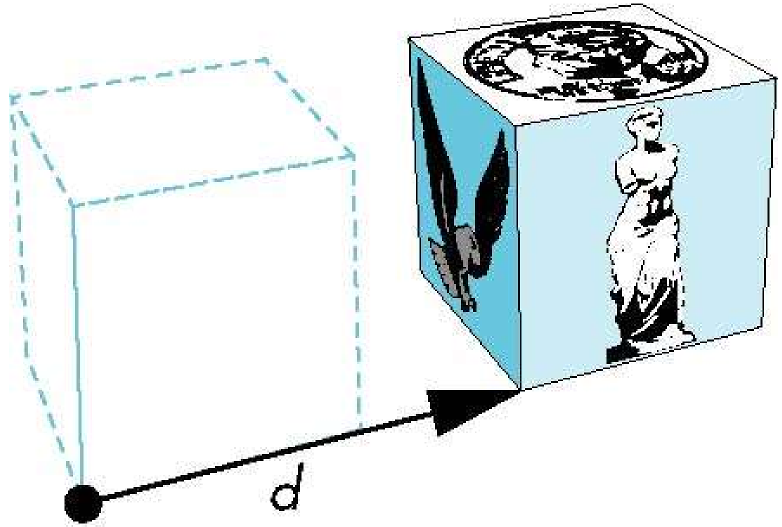
3D Transformations

- In general, three-dimensional transformation can be thought of as an extension of two-dimensional transformation.
- The basic principles of three-dimensional translation, scaling, shearing are the same as those of two-dimensional.
- However, three-dimensional rotation is a bit more complicated.

3D Translation

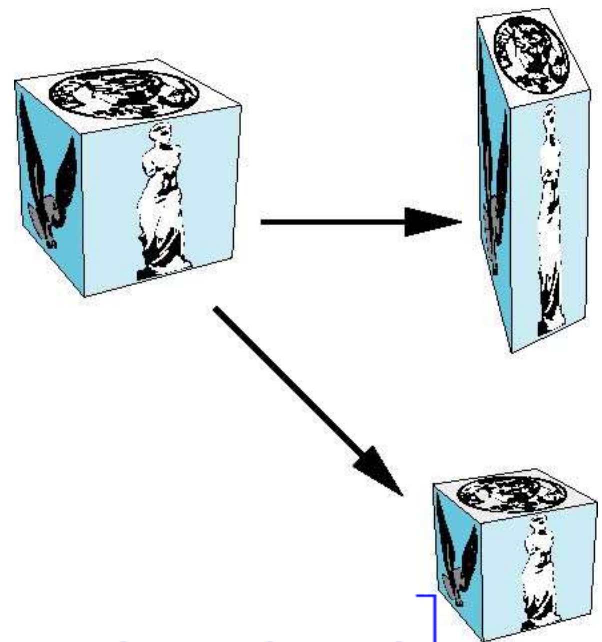
$$p' = p + d \qquad p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad p' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \qquad d = \begin{bmatrix} dx \\ dy \\ dz \\ 0 \end{bmatrix}$$



$$p' = Tp \qquad T = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -dx \\ 0 & 1 & 0 & -dy \\ 0 & 0 & 1 & -dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

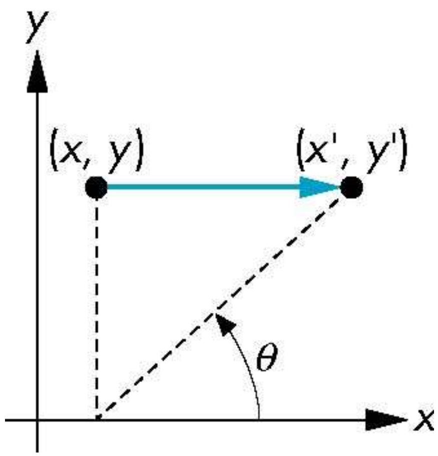
3D Scale

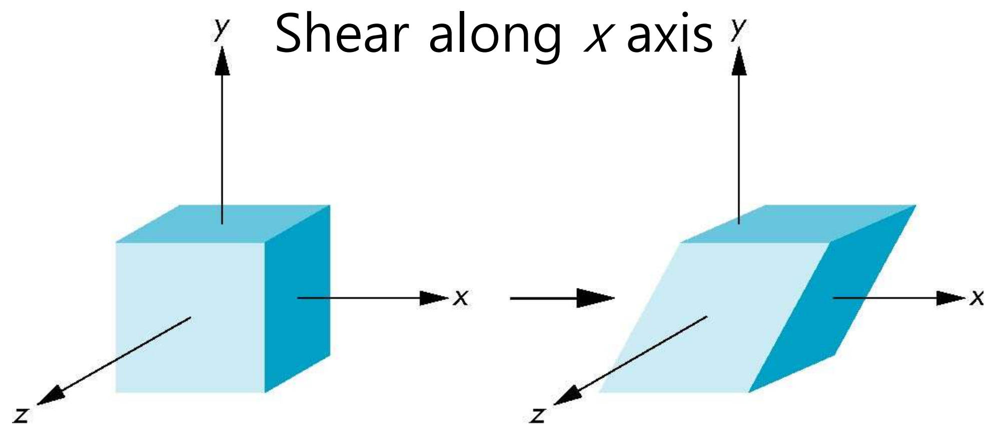
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$p' = Sp \qquad S = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} \frac{1}{sx} & 0 & 0 & 0 \\ 0 & \frac{1}{sy} & 0 & 0 \\ 0 & 0 & \frac{1}{sz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Shear





x′ = x + y cot θ
y′ = y
z′ = z

$$\mathbf{H}_{xy}(\theta) = \begin{bmatrix} 1 & \cot\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tan\theta = \frac{y}{x'-x} \Rightarrow \cot\theta = \frac{x'-x}{y}$$

$$R^{-1}(\theta) = R(-\theta)$$

$$R^{-1}(\theta) = R^{T}(\theta)$$

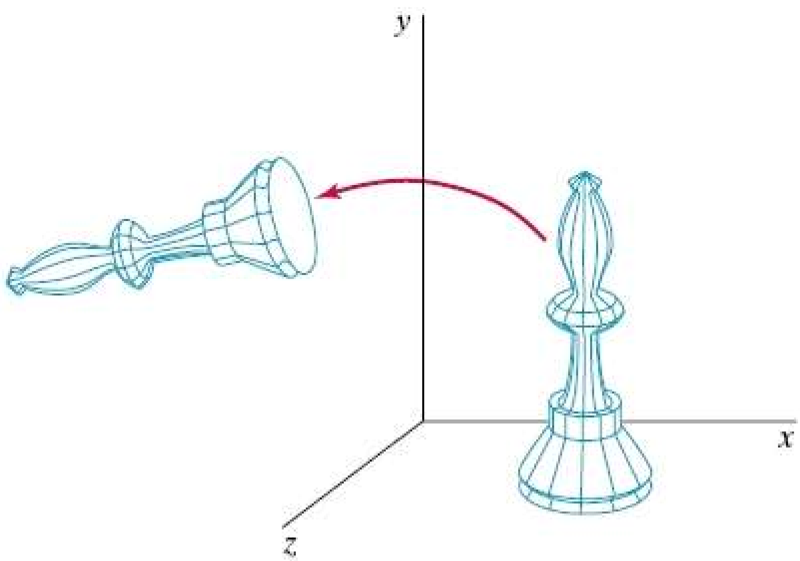
3D Rotation

- 3D rotation in Z-axis

  $x' = x\cos\theta - y\sin\theta$

  $y' = x\sin\theta + y\cos\theta$

  $z' = z$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_Z(\theta) \cdot P$$

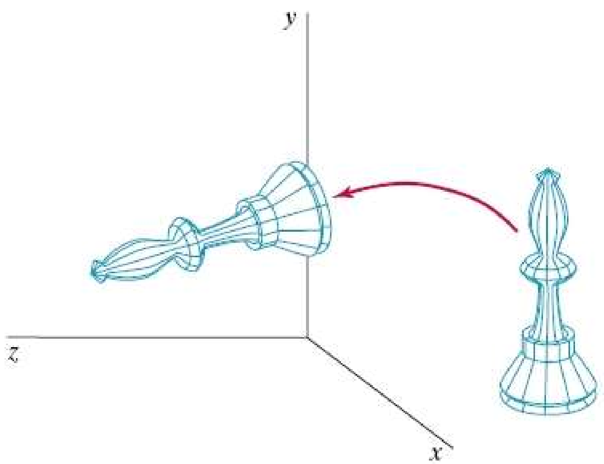
3D Rotation

- 3D rotation in X-axis

  $y' = y\cos\theta - z\sin\theta$

  $z' = y\sin\theta + z\cos\theta$

  $x' = x$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_X(\theta) \cdot P$$

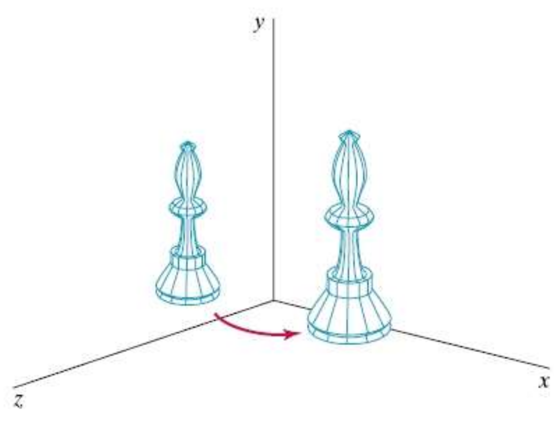
3D Rotation

- 3D rotation in Y-axis

  x′ = x cosθ + z sinθ

  z′ = -x sinθ + z cosθ

  y′ = y



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

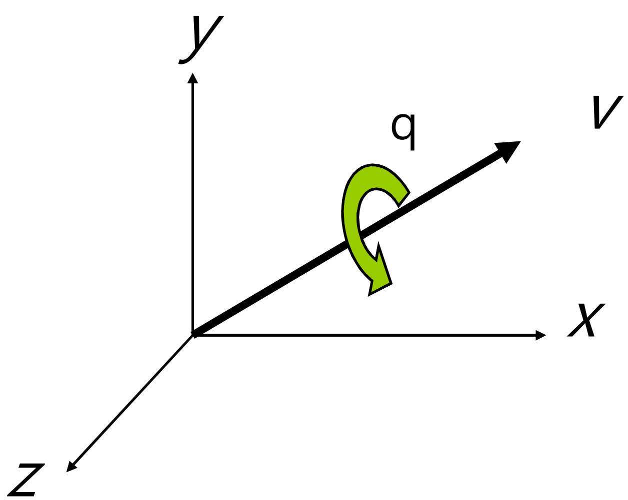
$$P' = R_Y(\theta) \cdot P$$

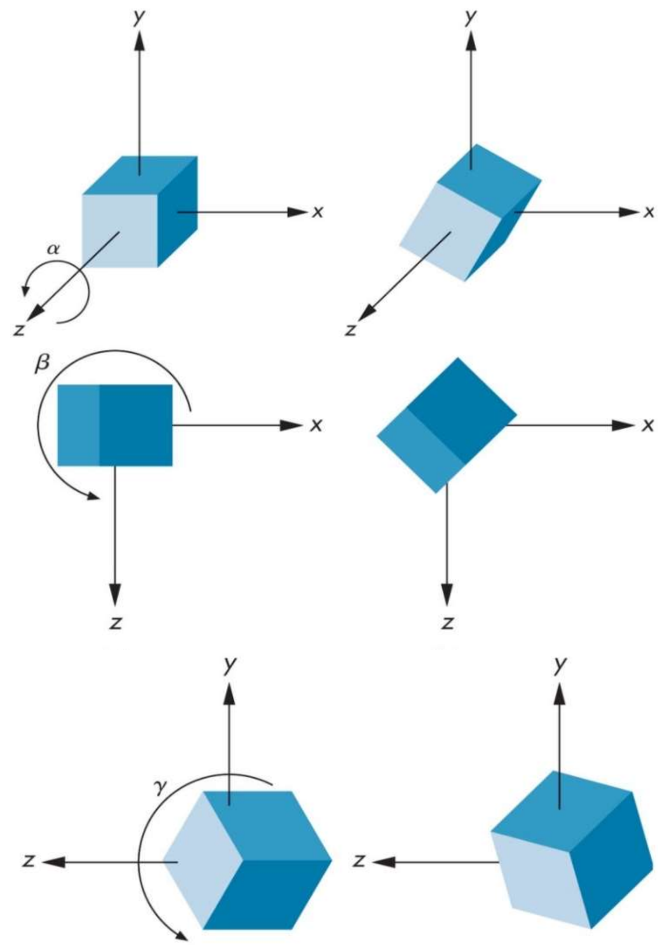
3D Rotation about the Origin

- A rotation by θ about an arbitrary axis can be decomposed into the concatenation of rotations about the x, y, and z axes.

  $$\mathbf{R}(\theta) = \mathbf{R}_Z(\theta_Z)\mathbf{R}_Y(\theta_Y)\mathbf{R}_X(\theta_X)$$

  θ_X, θ_Y, θ_Z are called the Euler angles.

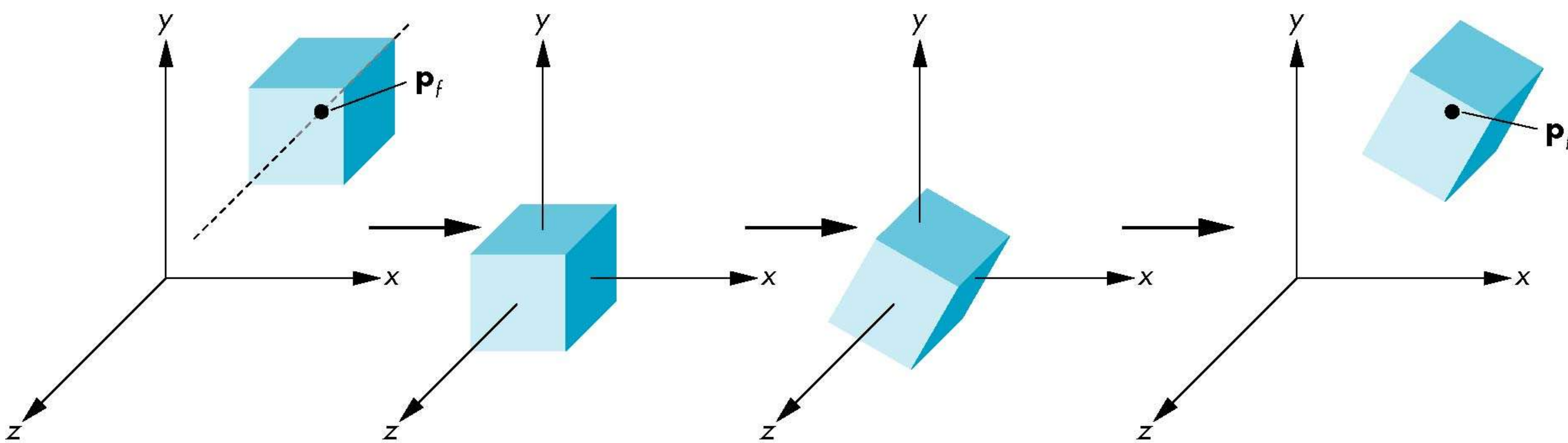




Rotation About a Pivot other than the Origin

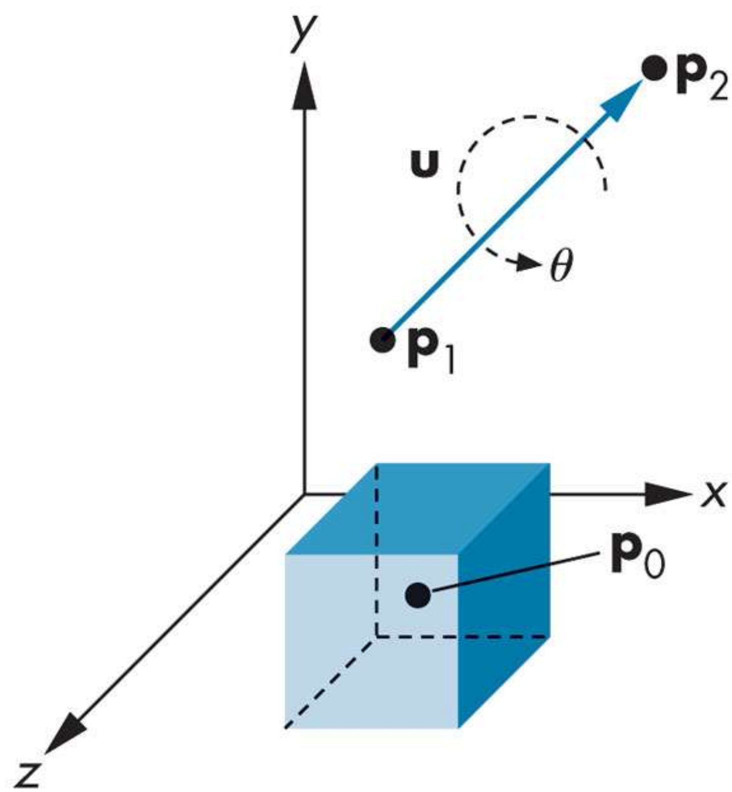
- Move fixed point to origin, rotate, and then move fixed point back.
- **M** = **T**(p_f) **R**$_Z$ (θ) **T**($-p_f$)

$$M = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & x_f - x_f\cos\theta + y_f\sin\theta \\ \sin\theta & \cos\theta & 0 & y_f - x_f\sin\theta - y_f\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3D Rotation about an Arbitrary Axis

- Move P_0 to the origin.
- Rotate twice to align the arbitrary axis u with the Z-axis.
- Rotate by θ in Z-axis.
- Undo two rotations (undo alignment).
- Move back to P_0.



$$M = T(P_0) R_x(-\theta_x) R_y(-\theta_y) R_z(\theta) R_y(\theta_y) R_x(\theta_x) T(-P_0)$$

3D Rotation about an Arbitrary Axis

- The translation matrix, $T(-P_0)$

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

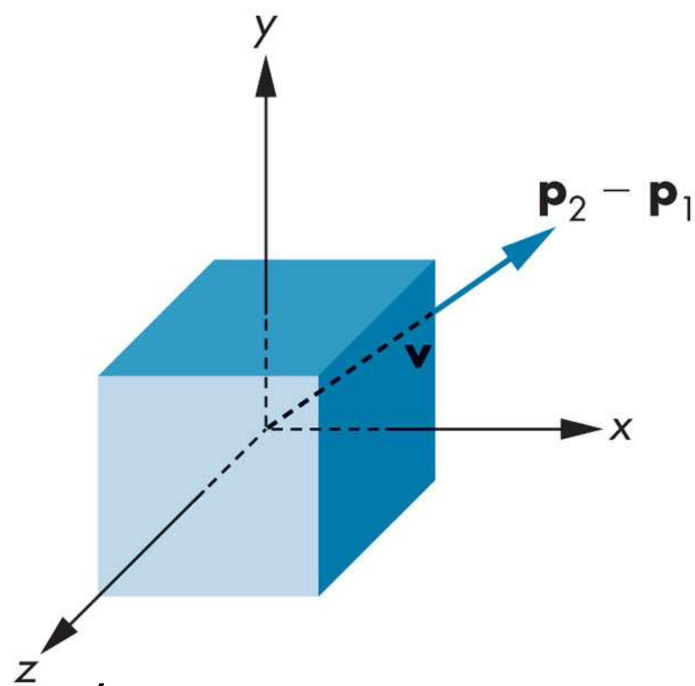
3D Rotation about an Arbitrary Axis

- The rotation-axis vector

$$u = P_2 - P_1$$
$$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

- Normalize u:

$$v = \frac{u}{\|u\|} = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$



- Rotate along x-axis until *v* hits *xz-plane*
- Rotate along y-axis until *v* hits *z-axis*

3D Rotation about an Arbitrary Axis

- Find θ_x and θ_y

  $v = (\alpha_x, \alpha_y, \alpha_z)$

  $\alpha_x^2 + \alpha_y^2 + \alpha_z^2 = 1$

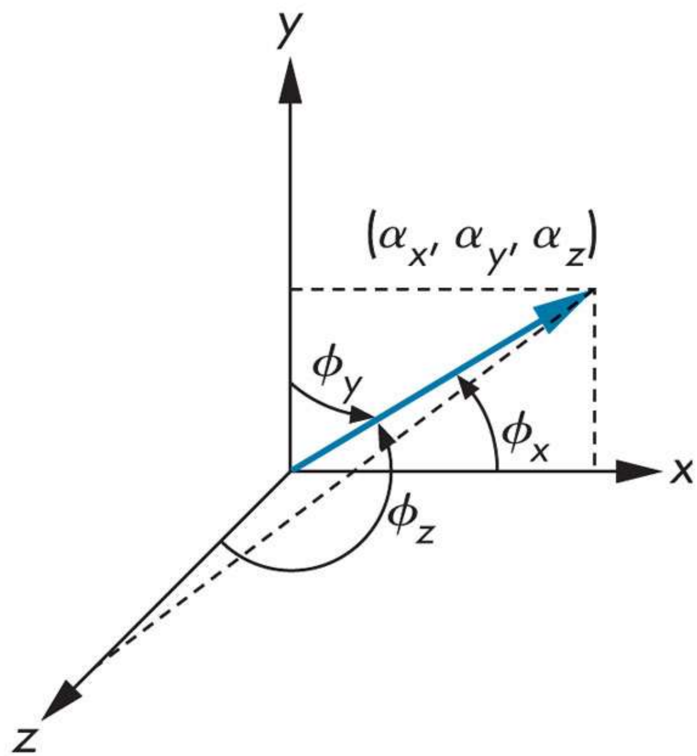
- Direction cosines:

$$\cos\phi_x = \alpha_x$$

$$\cos\phi_y = \alpha_y$$

$$\cos\phi_z = \alpha_z$$

$$\cos^2\phi_x + \cos^2\phi_y + \cos^2\phi_z = 1$$

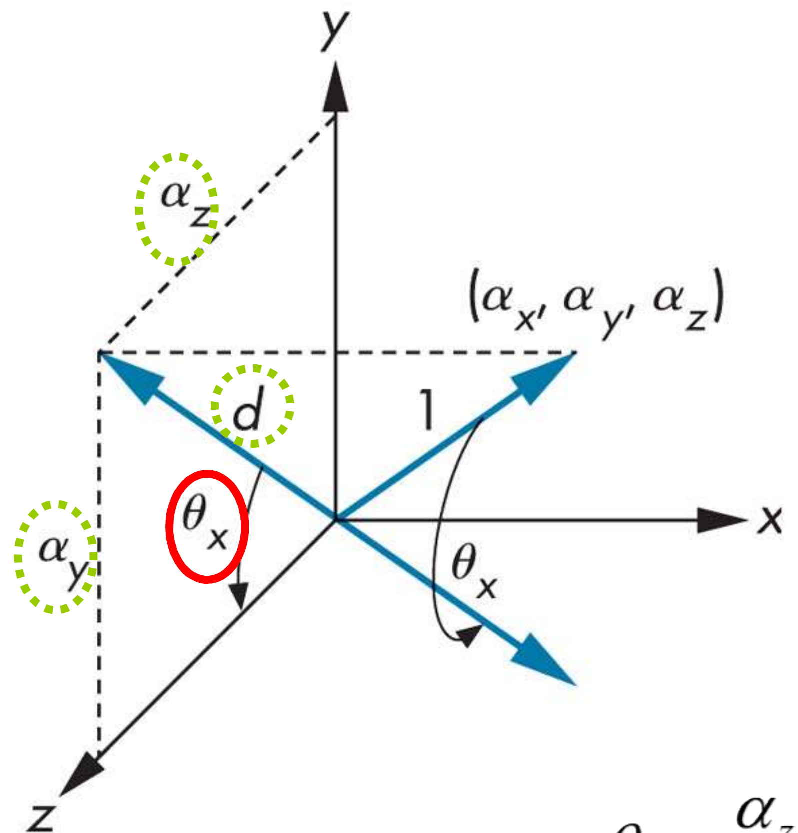


3D Rotation about an Arbitrary Axis

- Compute x-rotation θ_x

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha_z}{d} & -\frac{\alpha_y}{d} & 0 \\ 0 & \frac{\alpha_y}{d} & \frac{\alpha_z}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$



$$\cos\theta_x = \frac{\alpha_z}{d}$$

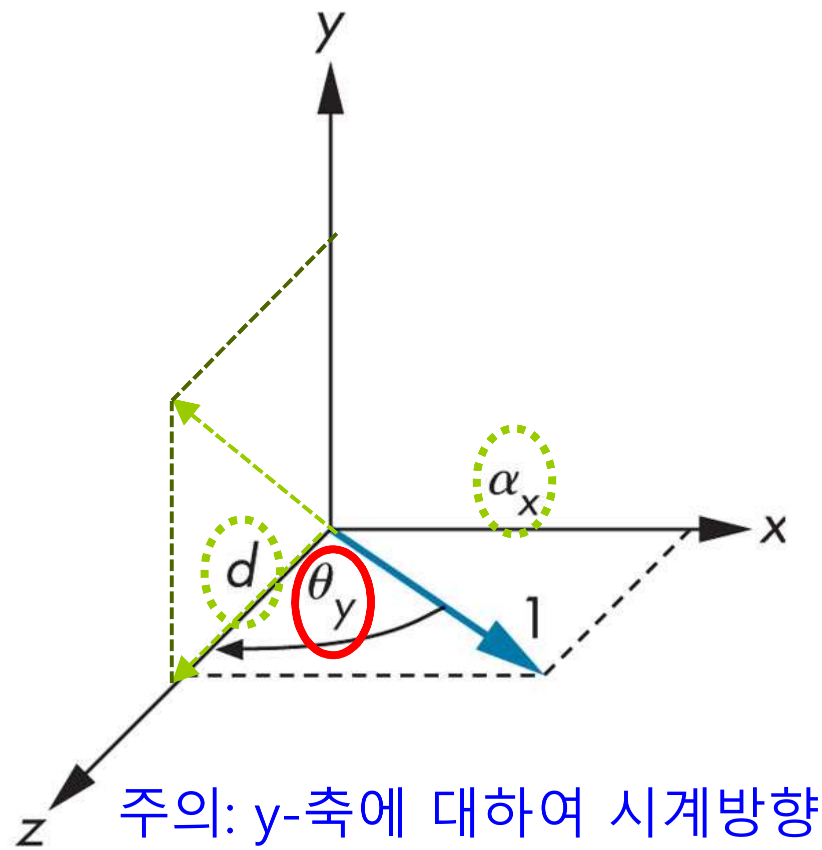
$$\sin\theta_x = \frac{\alpha_y}{d}$$

3D Rotation about an Arbitrary Axis

- Compute y-rotation θ_y

$$R_y(\theta_y) = \begin{bmatrix} d & 0 & -\alpha_x & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$



$$\cos\theta_y = d$$

$$\sin\theta_y = \alpha_x$$

3D Rotation about an Arbitrary Axis

- Rotation about the z axis

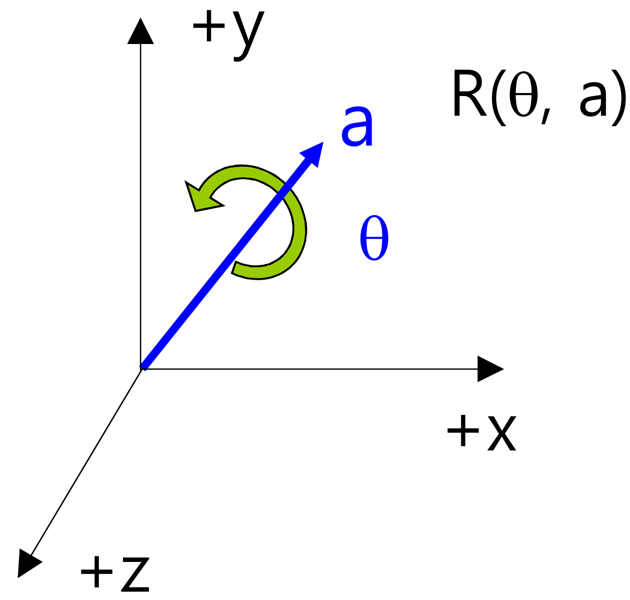
$$R_z(\theta_z) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Undo alignment, $R_x(-\theta_x)R_y(-\theta_y)$
- Undo translation, $T(P_0)$

- $M = T(P_0)R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x)T(-P_0)$

3D Rotation about an Arbitrary Axis Using Rotation Vectors

- 3D rotation can be expressed as 4 numbers of one angle of rotation about an arbitrary axis (ax, ay, az).
- It consists of a unit vector a (x, y, z) representing an arbitrary axis of rotation and a value of θ (0~360 degrees) representing the rotation angle around the unit vector.
- 3D rotation vector



3D Rotation about an Arbitrary Axis

- From axis/angle, we make the following rotation matrix.

$$R = \mathrm{I}\cos\theta + \mathbf{Symmetric}\ (1-\cos\theta) + \mathbf{Skew}\ \sin\theta$$

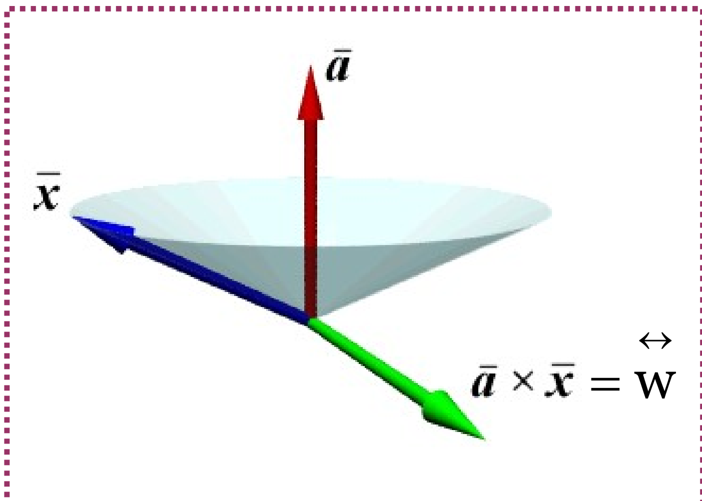
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cos\theta + \begin{bmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{bmatrix} (1-\cos\theta) + \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \sin\theta$$

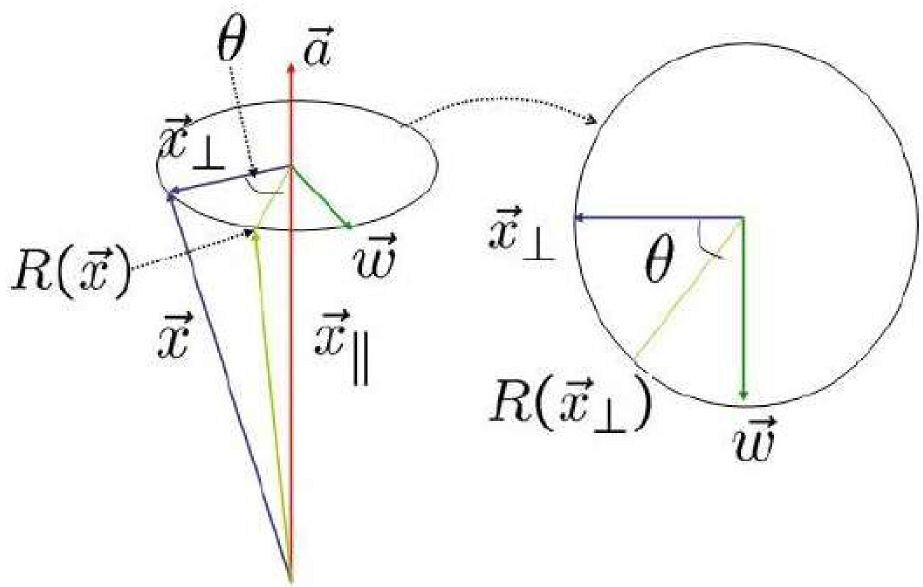
$$= \begin{bmatrix} a_x^2 + \cos\theta(1-a_x^2) & a_x a_y(1-\cos\theta) - a_z \sin\theta & a_x a_z(1-\cos\theta) + a_y \sin\theta \\ a_x a_y(1-\cos\theta) + a_z \sin\theta & a_y^2 + \cos\theta(1-a_y^2) & a_y a_z(1-\cos\theta) - a_x \sin\theta \\ a_x a_z(1-\cos\theta) - a_y \sin\theta & a_y a_z(1-\cos\theta) + a_x \sin\theta & a_z^2 + \cos\theta(1-a_z^2) \end{bmatrix}$$

3D Rotation as Vector Components

- 3D rotation by θ around the arbitrary axis a =[a_x, a_y, a_z]

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \left(\mathbf{Symmetric}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \right)(1-\cos\theta) + \mathbf{Skew}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \right)\sin\theta + \mathbf{I}\cos\theta \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

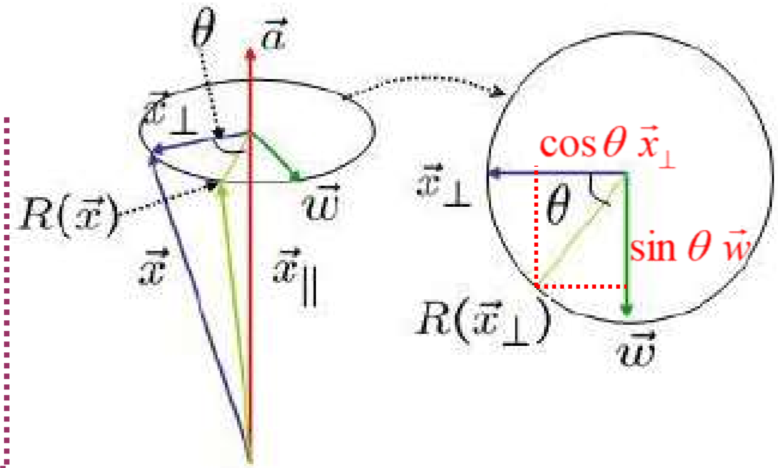




$$\left(R(\vec{x}_\perp)\bullet\vec{x}_\perp\right)\vec{x}_\perp = \cos\theta\,\vec{x}_\perp$$

3D Rotation as Vector Components

$$\begin{aligned}\vec{w} &= \vec{a}\times\vec{x}_\perp\\ &= \vec{a}\times(\vec{x}-\vec{x}_{||})\\ &= (\vec{a}\times\vec{x})-(\vec{a}\times\vec{x}_{||})\\ &= \vec{a}\times\vec{x}\end{aligned}$$



$$R(\vec{x}_\perp) = \cos\theta\,\vec{x}_\perp + \sin\theta\,\vec{w}$$

$$\begin{aligned}\vec{x}_{||} &= (\vec{a}\cdot\vec{x})\vec{a}\\ \vec{x}_\perp &= \vec{x}-\vec{x}_{||} = \vec{x}-(\vec{a}\cdot\vec{x})\vec{a}\end{aligned}$$

$$\begin{aligned}R(\vec{x}) &= R(\vec{x}_{||}) + R(\vec{x}_\perp)\\ &= R(\vec{x}_{||}) + \cos\theta\,\vec{x}_\perp + \sin\theta\,\vec{w}\\ &= (\vec{a}\cdot\vec{x})\vec{a} + \cos\theta(\vec{x}-(\vec{a}\cdot\vec{x})\vec{a}) + \sin\theta\,\vec{w}\\ &= \cos\theta\,\vec{x} + (1-\cos\theta)(\vec{a}\cdot\vec{x})\vec{a} + \sin\theta(\vec{a}\times\vec{x})\end{aligned}$$

Symmetric

Skew

3D Rotation as Vector Components

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \left(\mathbf{Symmetric}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right)(1-\cos\theta) + \mathbf{Skew}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right)\sin\theta + \mathbf{I}\cos\theta \right)\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- The vector *a* specifies the axis of rotation. This axis vector must be normalized.
- The rotation angle is given by q.
- The basic idea is that *any rotation can be decomposed into weighted contributions from three different vectors*.

3D Rotation as Vector Components

- *The symmetric matrix of a vector* generates a vector in the direction of the axis.
- The symmetric matrix is composed of the outer product of a row vector and an column vector of the same value.

$$\text{Symmetric}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right) = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\begin{bmatrix} a_x & a_y & a_z \end{bmatrix} = \begin{bmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{bmatrix}$$

$$\text{Symmetric}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \bar{a}(\bar{a} \cdot \bar{x})$$

3D Rotation as Vector Components

- *Skew symmetric matrix of a vector* generates a vector that is perpendicular to both the axis and it's input vector.

$$\mathrm{Skew}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\mathrm{Skew}(\overline{a})\overline{x} = \overline{a} \times \overline{x}$$

3D Rotation as Vector Components

- First, consider a rotation by 0. :

$$Rotate\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, 0\right) = \begin{bmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{bmatrix}(1-1) + \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}0 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- For instance, a rotation about the x-axis:

$$Rotate\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \theta\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(1-\cos\theta) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\sin\theta + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cos\theta$$

$$Rotate\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \theta\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

3D Rotation as Vector Components

- For instance, a rotation about the y-axis:

$$Rotate\left(\begin{bmatrix}0\\1\\0\end{bmatrix},\theta\right)=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}(1-\cos\theta)+\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}\sin\theta+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cos\theta$$

$$Rotate\left(\begin{bmatrix}0\\1\\0\end{bmatrix},\theta\right)=\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix}$$

- For instance, a rotation about the z-axis:

$$Rotate\left(\begin{bmatrix}0\\0\\1\end{bmatrix},\theta\right)=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}(1-\cos\theta)+\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\sin\theta+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cos\theta$$

$$Rotate\left(\begin{bmatrix}0\\0\\1\end{bmatrix},\theta\right)=\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$$