

Transformation

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Geometric Objects

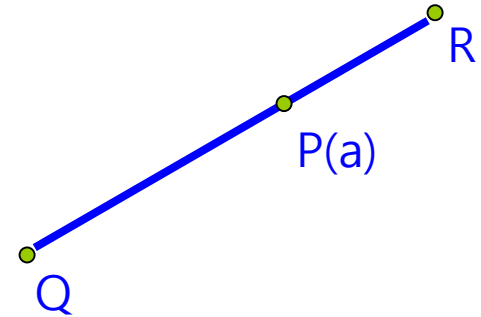
□ Line

- 2 points: R, Q

- $v = R - Q$

$$P = Q + \alpha v = Q + \alpha(R - Q) = \alpha R + (1 - \alpha)Q$$

- $P = \alpha_1 R + \alpha_2 Q$ where $\alpha_1 + \alpha_2 = 1$ (affine sum)

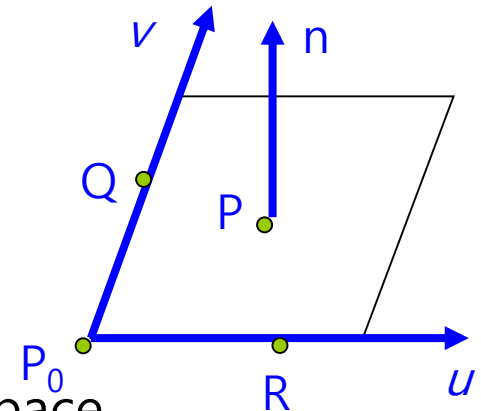


□ Plane

- 3 points: P_0 , Q, R

- $T(\alpha, \beta) = P + \alpha u + \beta v$

- $n \cdot (P - P_0) = 0$ where $n = u \times v$



□ 3D objects

- It is a set of vertices in three dimensional space.
- It is described by the surface, and is hollow.
- It can be composed of convex polygons.
- An arbitrary polygon is divided into triangular polygons, i.e., tessellate.

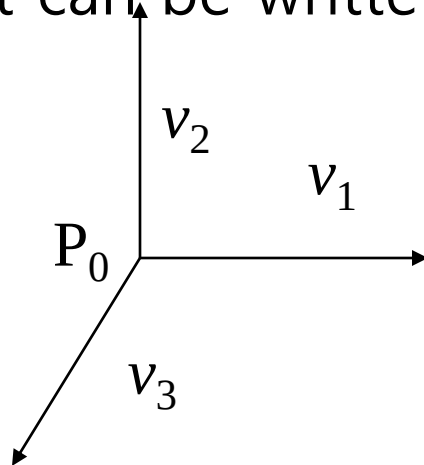
Coordinate Systems

- Consider a basis, $v_1, v_2, \dots, v_n$
- Any vector v can be written as $v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$
- The list of scalars $\{a_1, a_2, \dots, a_n\}$ is the representation of v with respect to the given basis.
- We can write the representation as a row or column array of scalars.

$$\mathbf{a} = [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_n]^T = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Frames

- The affine space contains points.
- If we work in an affine space we can add the origin to the basis vectors to form a **frame**.
- Frame: (P_0, v_1, v_2, v_3)
- Within this frame, every vector can be written as:
$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$
- Every point can be written as: $P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$



$$\mathbf{v} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ 0]^T$$
$$\mathbf{p} = [\beta_1 \ \beta_2 \ \beta_3 \ 1]^T$$

Change of Coordinate Systems

- Consider two representations of the same vector, v , with respect to two different bases : $\{v_1, v_2, v_3\}$, $\{u_1, u_2, u_3\}$

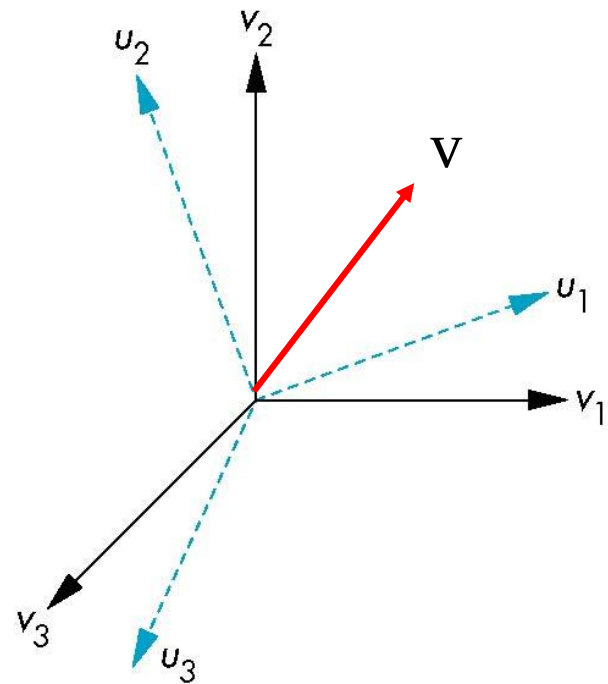
$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$

$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$

$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$



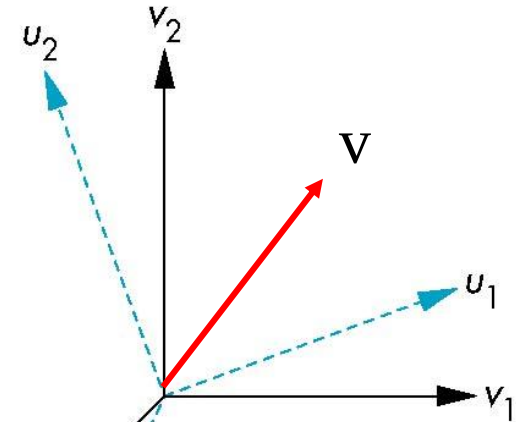
Change of Coordinate Systems

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}}_{\mathbf{a}^T} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$v = \underbrace{\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix}}_{\mathbf{b}^T} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

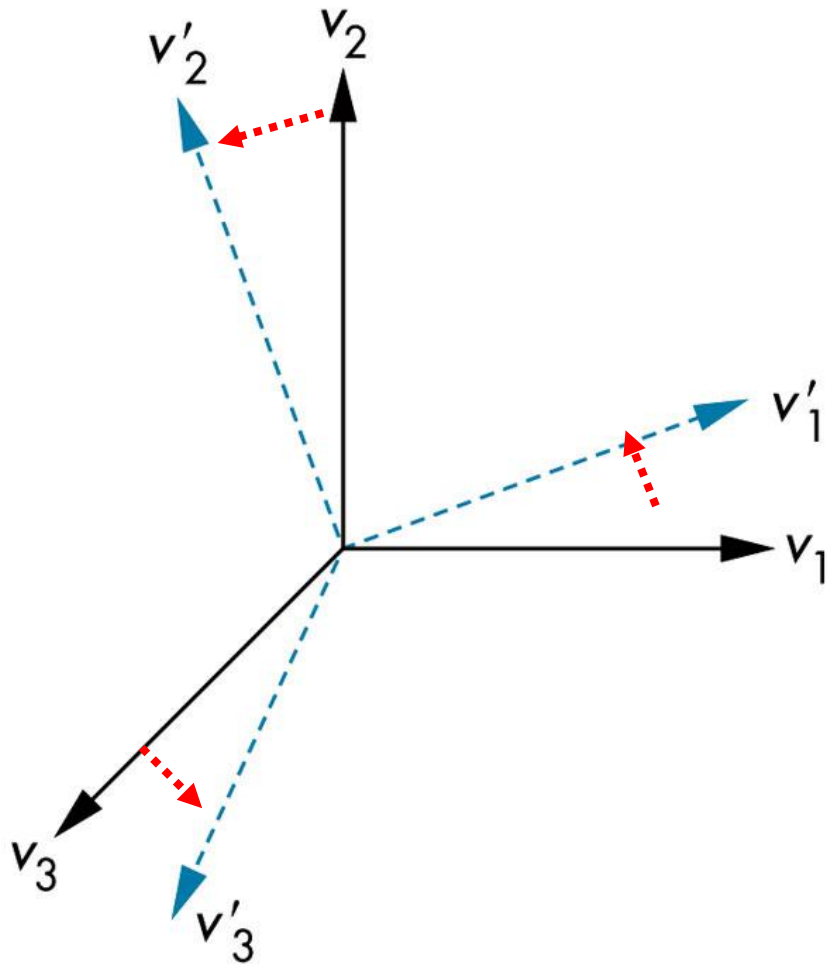
$$\underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}}_{\mathbf{a}^T} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\mathbf{v}} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix}}_{\mathbf{b}^T} \underbrace{\mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{\mathbf{a}} = M^T \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{\mathbf{b}} \quad \therefore \mathbf{a} = M^T \mathbf{b} \quad \therefore \mathbf{b} = (M^T)^{-1} \mathbf{a}$$



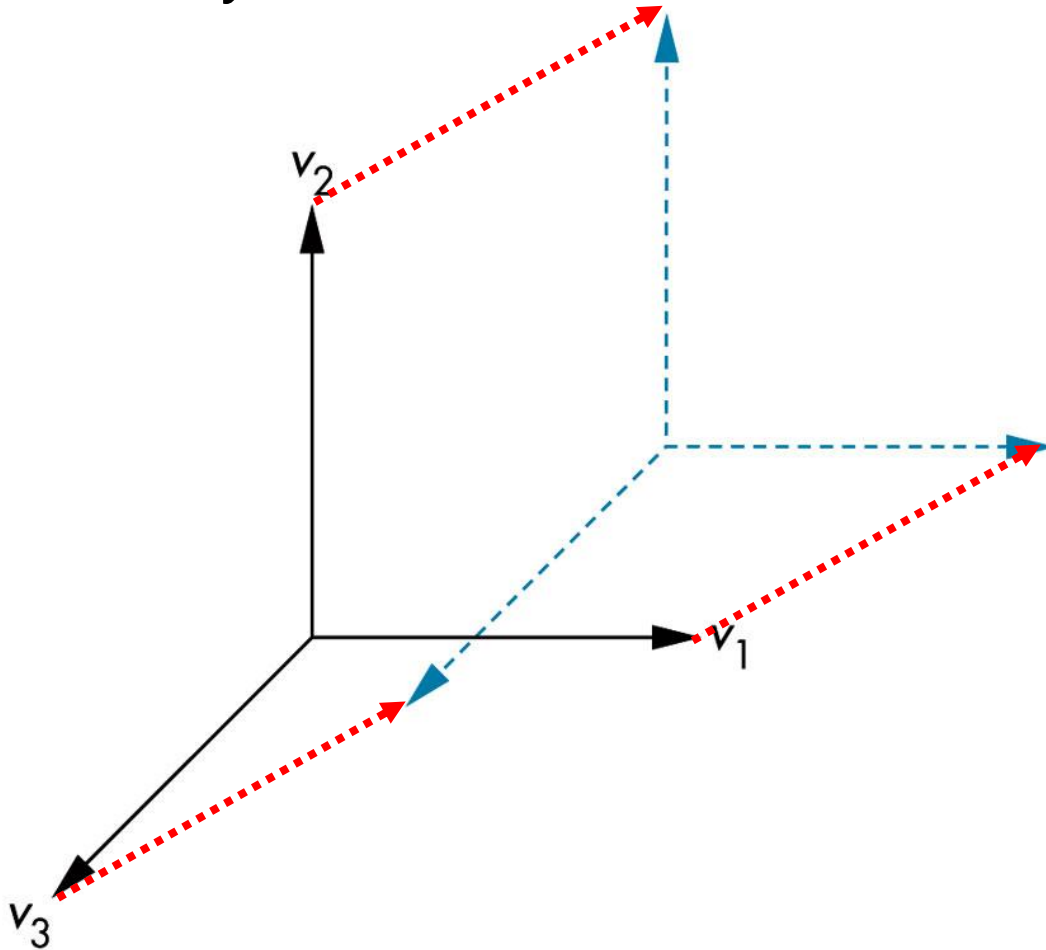
Rotation and Scaling of a Basis

- The rotation and scaling transformation can be represented by the basis vectors.



Translation of a Basis

- However, a simple translation of the origin is not represented by a 3×3 matrix.

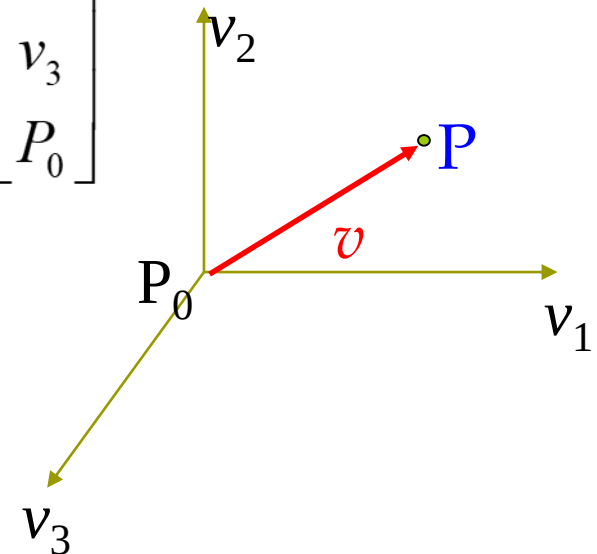


Homogeneous Coordinates

$$\mathbf{vector} \ v = \sum \alpha_i v_i = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$\mathbf{point} \ P = P_0 + \sum \alpha_i v_i = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$P = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix}, v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 0 \end{bmatrix}$$



Change of Frames

□ Consider two frames (P_0, v_1, v_2, v_3) (Q_0, u_1, u_2, u_3)

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$

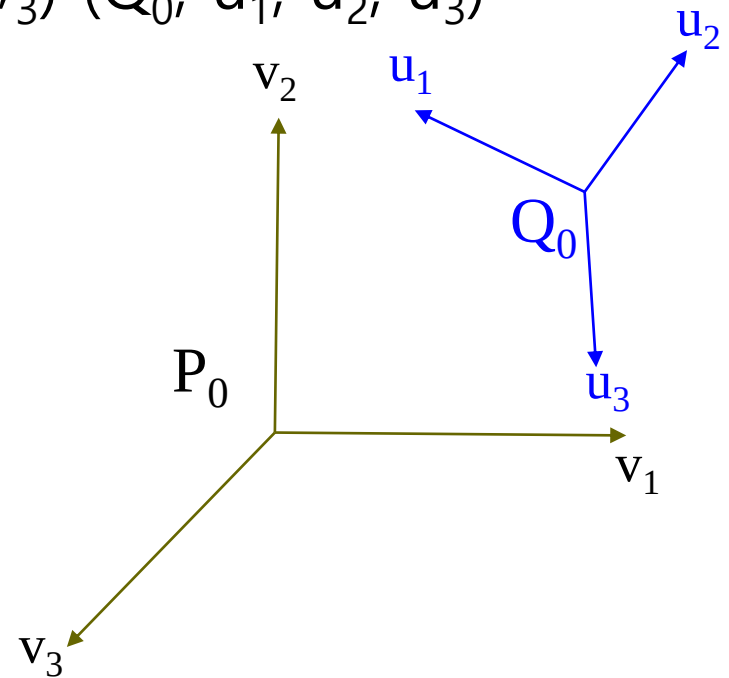
$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$

$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$

$$Q_0 = \gamma_{41}v_1 + \gamma_{42}v_2 + \gamma_{43}v_3 + P_0$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$



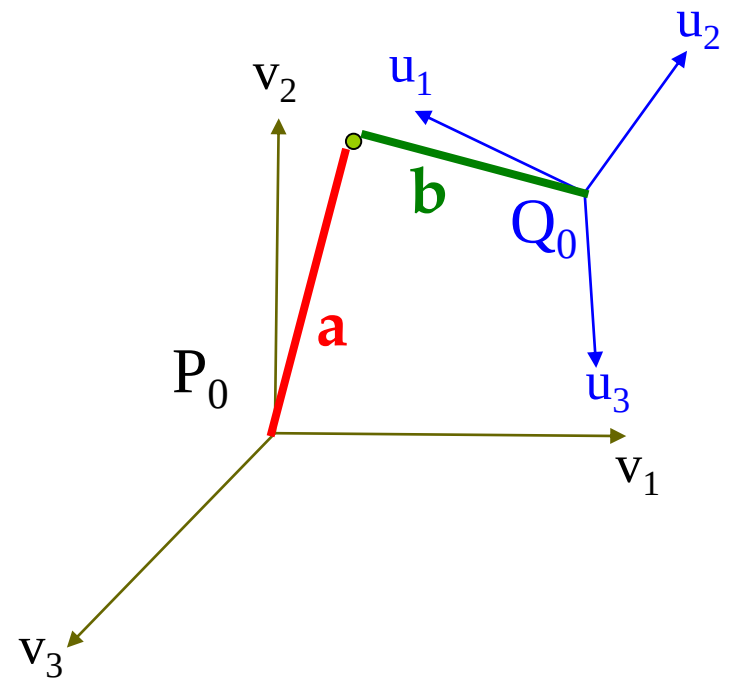
Change of Frames

- Within the two frames (P_0, v_1, v_2, v_3) (Q_0, u_1, u_2, u_3) any point and vector has a representation of the same form

$$\begin{aligned} b^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} &= b^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = a^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} \\ M^T &= \begin{bmatrix} \gamma_{11} & \gamma_{21} & \gamma_{31} & \gamma_{41} \\ \gamma_{12} & \gamma_{22} & \gamma_{32} & \gamma_{42} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & \gamma_{43} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\therefore a = M^T b$$

$$\therefore b = (M^T)^{-1} a$$

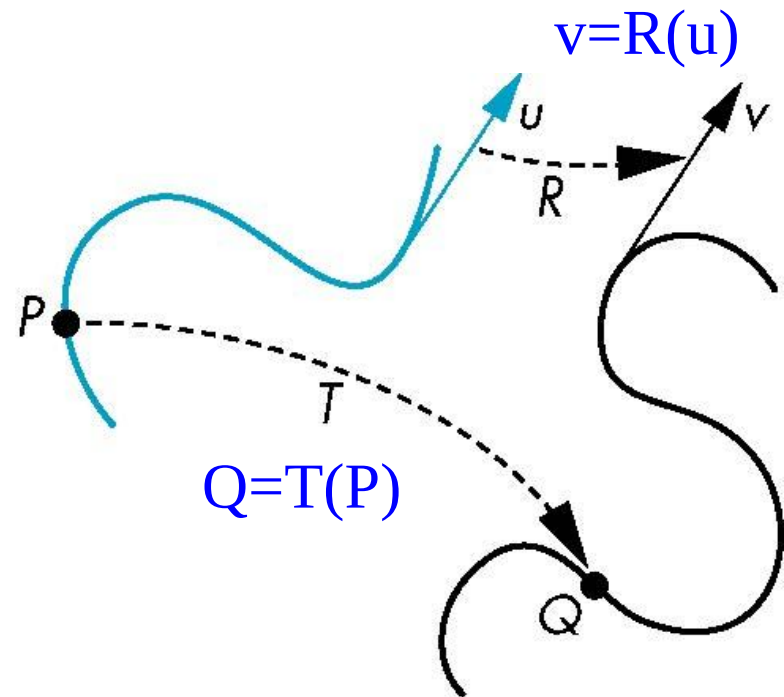


General Transformations

- A transformation maps points to other points and/or vectors to other vectors

$$q = f(p)$$

$$v = f(u)$$



Affine Transformations

- The affine transformation maintains collinearity.
 - That is, every affine transformation preserves lines. All points on a line exist on the transformed line.
- Also, it maintains the ratio of distance.
 - That is, the midpoint of a line is located at the midpoint of the transformed line segment.
- $P' = f(P)$
- $P' = f(\alpha P_1 + \beta P_2) = \alpha f(P_1) + \beta f(P_2)$

Affine Transformation

- Most transformation in computer graphics are affine transformation. Affine transformation include **translation, rotation, scaling, shearing**.
- The transformed point $P' (x', y', z')$ can be expressed as a linear combination of the original point $P (x, y, z)$, i.e.

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Affine Transformation

- The transformed point P' (x' , y' , z') can be expressed as a linear combination of the original point P (x , y , z), i.e.,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_{11}x + \alpha_{12}y + \alpha_{13} \\ \alpha_{21}x + \alpha_{22}y + \alpha_{23} \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Geometric Transformation

- ❑ Geometric transformation refers to a function that transforms a group of points describing a geometric object to new points.
- ❑ At this time, the points are transformed to a new position while maintaining the relationship between the vertices of the objects.
- ❑ Basic transformation
 - Translation
 - Rotation
 - Scaling

Unity Matrix Column-Major Order

- 2D transformation matrix, $\mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- If Point $\mathbf{p}$ is a column vector (Unity) :

$$\mathbf{p}' = \mathbf{M}\mathbf{p}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- If Point $\mathbf{p}$ is a row vector:

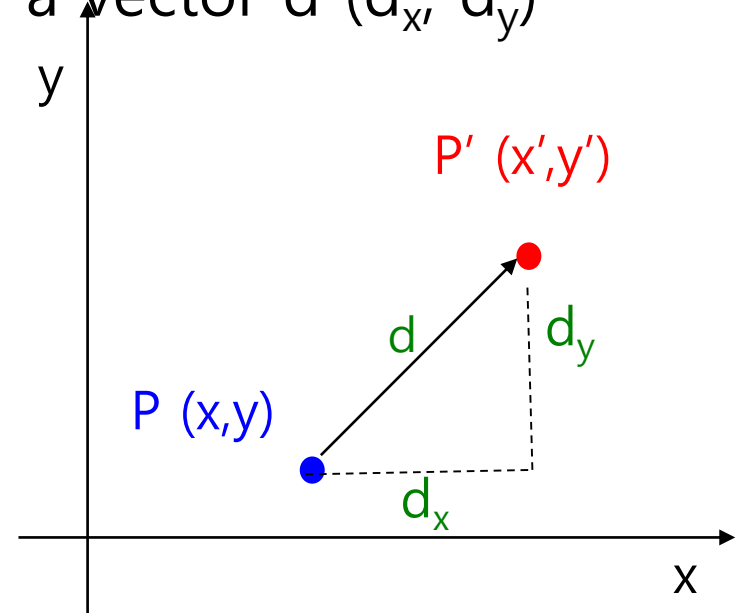
$$\mathbf{p}' = \mathbf{p}\mathbf{M}^T$$
$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

2D Translation□

- Translation moves a point $P(x, y)$ to a new location $P'(x', y')$
- Displacement determined by a vector $d (d_x, d_y)$

$$x' = x + d_x$$

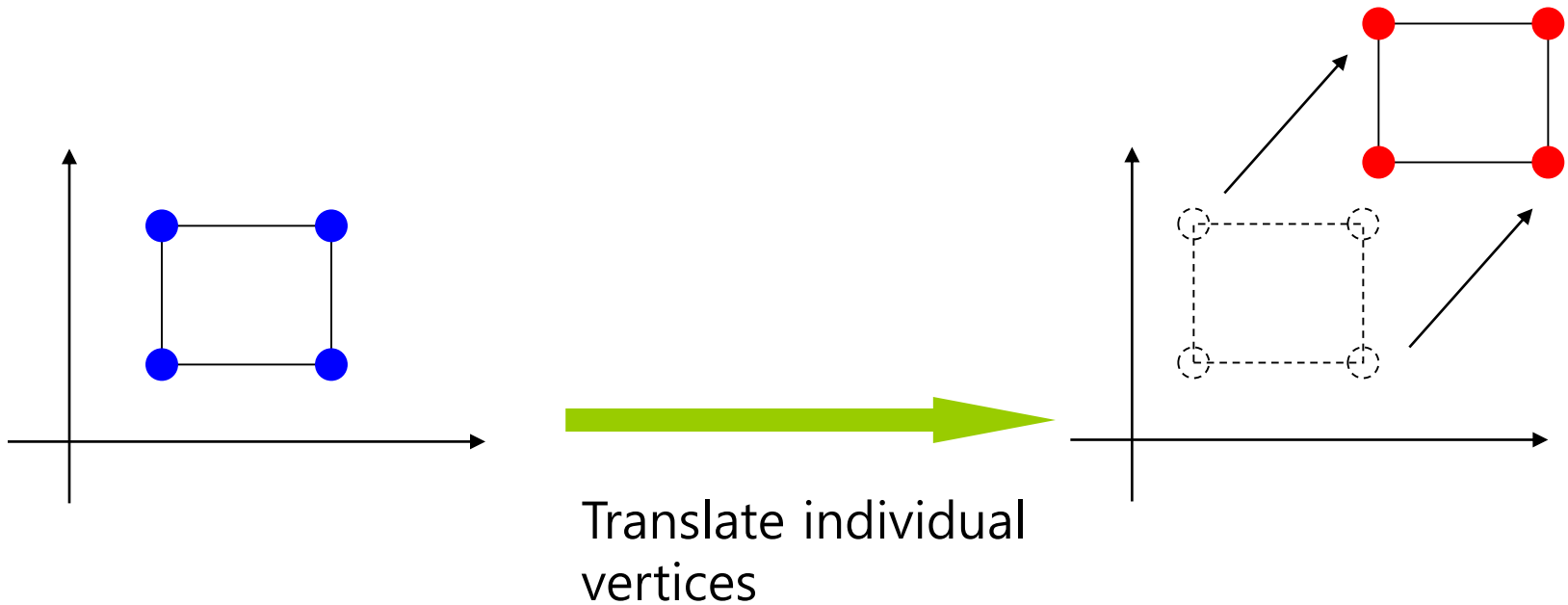
$$y' = y + d_y$$



$$P' = P + d \text{ where } P' = \begin{pmatrix} x' \\ y' \end{pmatrix} P = \begin{pmatrix} x \\ y \end{pmatrix} d = \begin{pmatrix} d_x \\ d_y \end{pmatrix}$$

2D Translation□

- What if you move an object with multiple vertices?



2D Translation

- Using the homogeneous coordinate representation in some frame

$$\mathbf{p} = [x \ y \ 1]^T$$

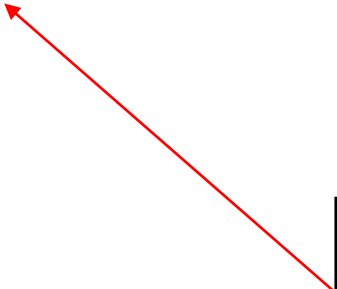
$$\mathbf{p}' = [x' \ y' \ 1]^T$$

$$\mathbf{d} = [dx \ dy \ 0]^T$$

- Hence $\mathbf{p}' = \mathbf{p} + \mathbf{d}$ or

$$x' = x + d_x$$

$$y' = y + d_y$$



Note that this expression is in four dimensions and expresses point = vector + point

2D Translation

- We can also express 2D translation using a 3 x 3 matrix $\mathbf{T}$ in homogeneous coordinates:

$\mathbf{p}' = \mathbf{T}\mathbf{p}$ where

$$\mathbf{T} = \mathbf{T}(d_x, d_y) = \begin{pmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{pmatrix}$$

- This form is better for implementation because all affine transformations can be expressed this way and multiple transformations can be concatenated together.

2D Translation

□ 2D translation

$$x' = x + d_x$$

$$y' = y + d_y$$

□ Inverse translation

$$x = x' - d_x$$

$$y = y' - d_y$$

□ Identity translation

$$x' = x + 0$$

$$y' = y + 0$$

2D Rotation

□ Rotation of a point $P(x,y)$ by θ about an origin $(0,0)$

$$x = r \cos(\phi) \quad y = r \sin(\phi)$$

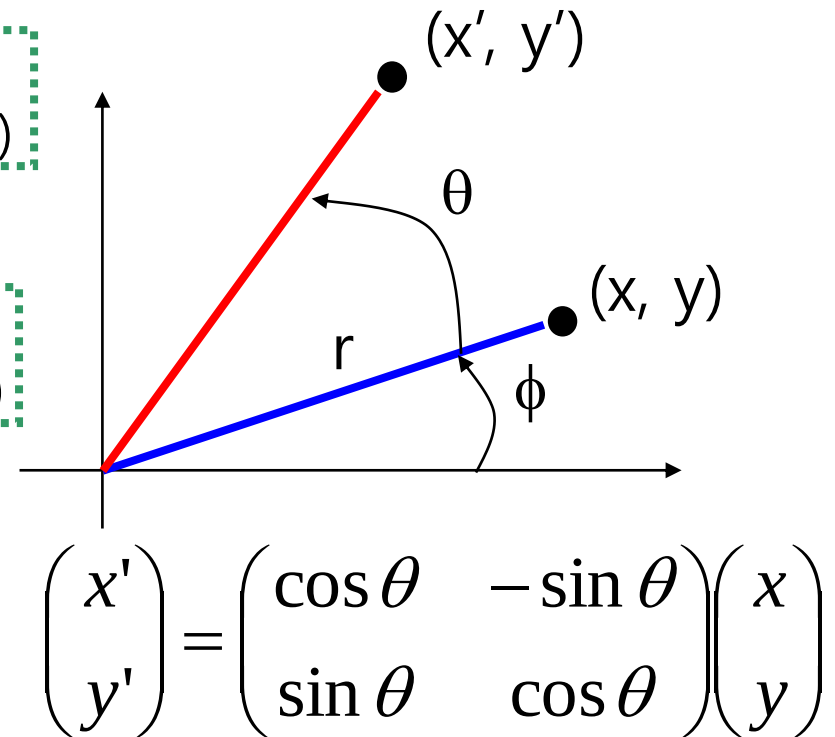
$$x' = r \cos(\phi + \theta) \quad y' = r \sin(\phi + \theta)$$

$$\begin{aligned} x' &= r \cos(\phi + \theta) \\ &= \boxed{r \cos(\phi)} \cos(\theta) - \boxed{r \sin(\phi)} \sin(\theta) \\ &= x \cos(\theta) - y \sin(\theta) \end{aligned}$$

$$\begin{aligned} y' &= r \sin(\phi + \theta) \\ &= \boxed{r \sin(\phi)} \cos(\theta) + \boxed{r \cos(\phi)} \sin(\theta) \\ &= y \cos(\theta) + x \sin(\theta) \end{aligned}$$

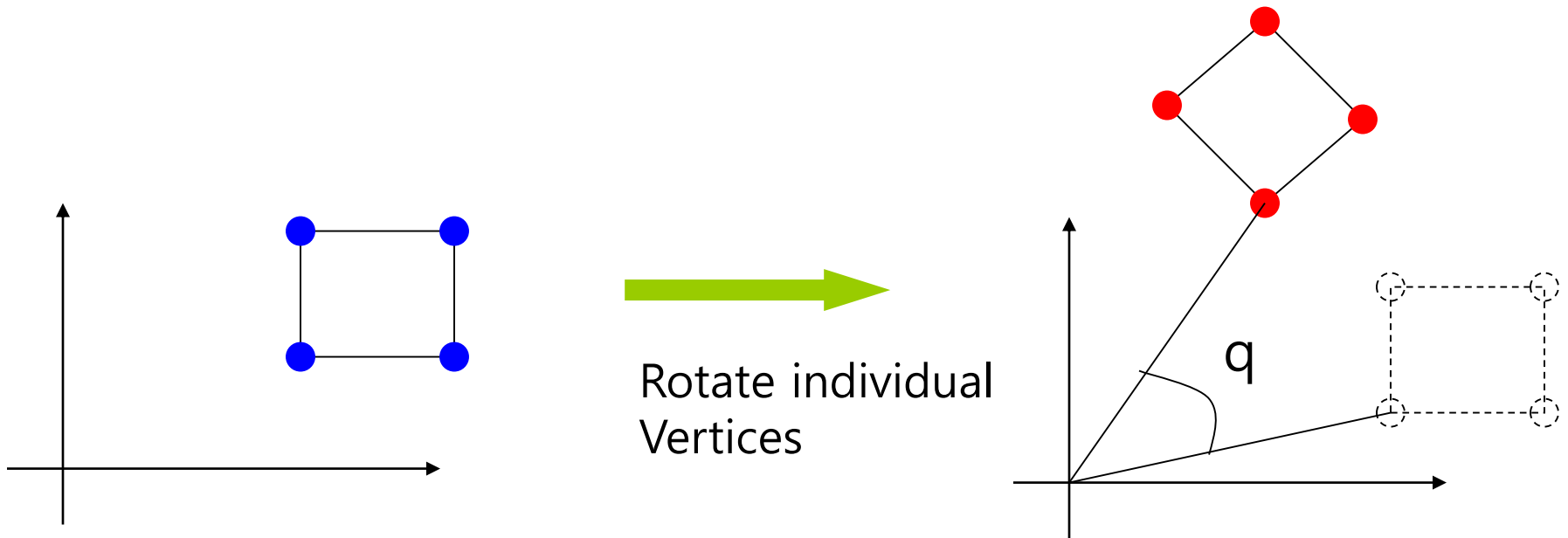
$$x' = x \cos(\theta) - y \sin(\theta)$$

$$y' = y \cos(\theta) + x \sin(\theta)$$



2D Rotation

- What if you rotate an object with multiple vertices?



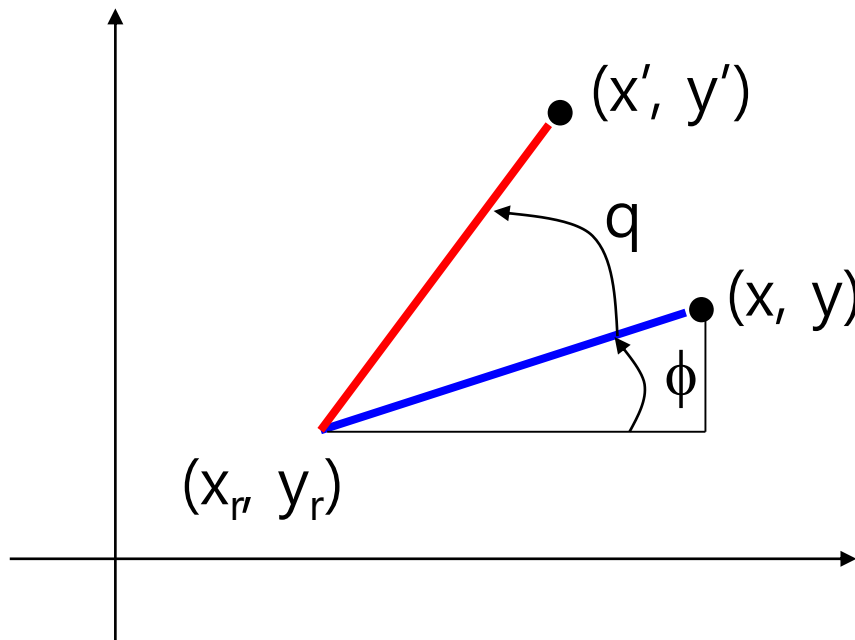
2D Rotation about an Arbitrary Pivot

- Rotation of a point $P(x, y)$ by θ about an arbitrary pivot point, (x_r, y_r) :

$$P' = R(\theta) P$$

$$x' = x_r + (x - x_r) \cos \theta - (y - y_r) \sin \theta$$

$$y' = y_r + (x - x_r) \sin \theta + (y - y_r) \cos \theta$$



2D Rotation

□ 2D rotation

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

□ Inverse rotation

$$R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

□ Identity rotation

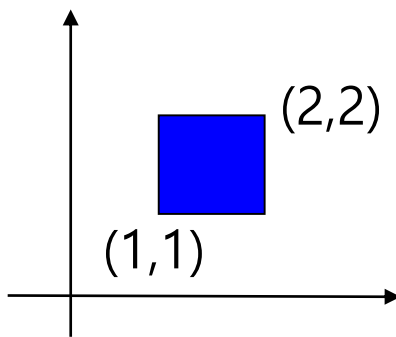
$$R_{\theta=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2D Scale

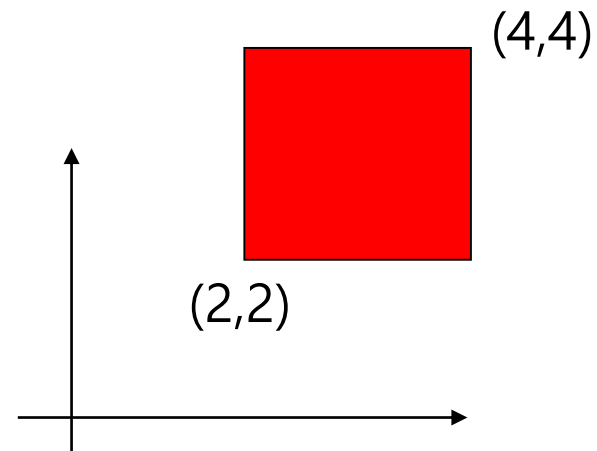
- Scaling makes an object larger or smaller by a scaling factor (s_x, s_y). This is affine non-rigid-body transformation. Scaling by 1 does not change an object.
- Scaling is done by an origin. Scaling changes not only the size of object, but also the position of object.

$$x' = x \cdot s_x$$

$$y' = y \cdot s_y$$



$$s_x = 2, s_y = 2$$



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale about an Arbitrary Pivot

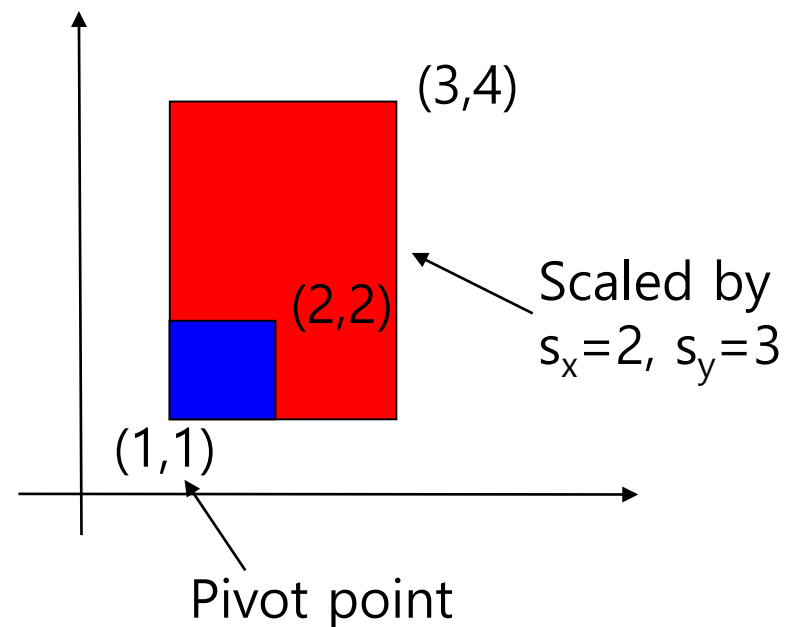
- Scale a point $P(x,y)$ by a scaling factor relative to an arbitrary pivot point, (x_f, y_f) : $P' = S(s_x, s_y) P$

$$x' = x_f + (x - x_f) s_x$$

$$y' = y_f + (y - y_f) s_y$$

$$x' = x s_x + x_f (1 - s_x)$$

$$y' = y s_y + y_f (1 - s_y)$$



2D Scale

□ 2D scale

$$S = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$$

□ Inverse scale

$$S^{-1} = \begin{pmatrix} 1/s_x & 0 \\ 0 & 1/s_y \end{pmatrix}$$

□ Identity scale

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2D Reflection (Mirror)

- Reflection is the transformation of an object in opposite direction with respect to a fixed point.
 - Reflection preserves angles and lengths.

- 2D reflection over x axis

$$x' = x$$

$$y' = -y$$

- 2D reflection over y axis

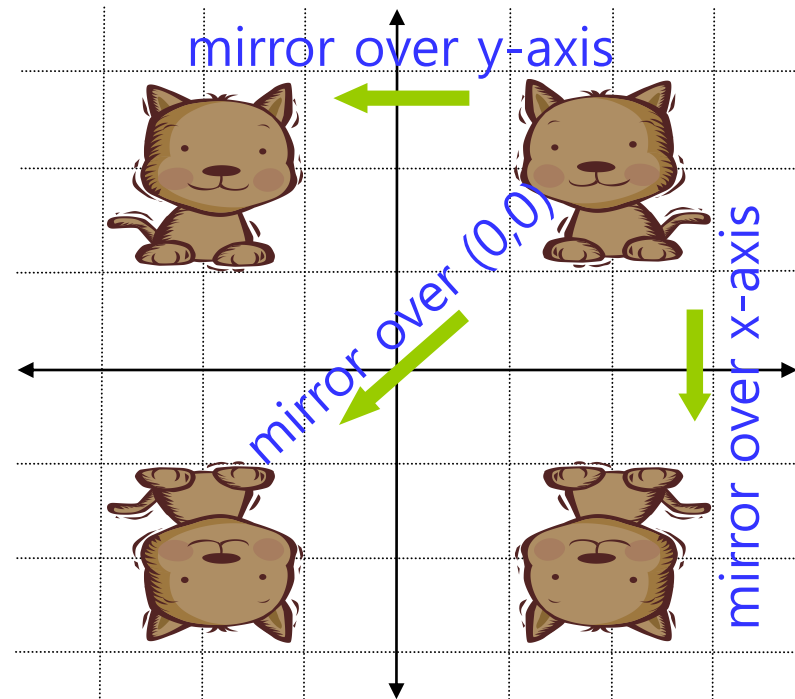
$$x' = -x$$

$$y' = y$$

- 2D reflection over (0,0)

$$x' = -x$$

$$y' = -y$$



2D Reflection (Mirror)

- 2D reflection over a line, $y = x$

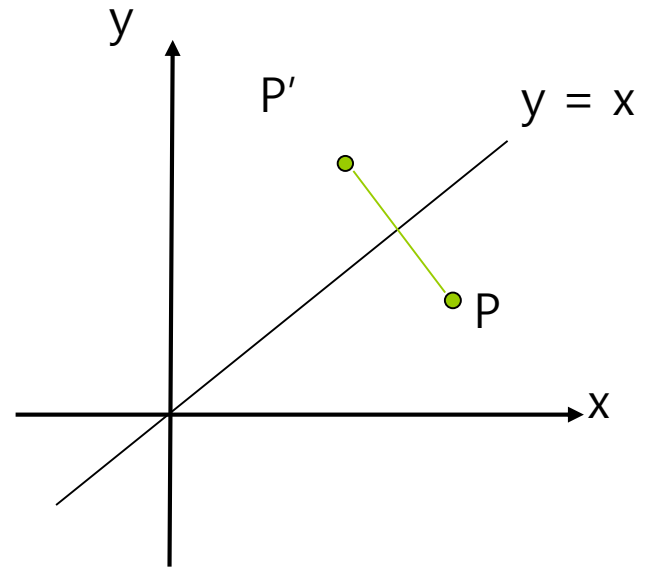
$$x' = y$$

$$y' = x$$

- 2D reflection over a line, $y = -x$

$$x' = -y$$

$$y' = -x$$



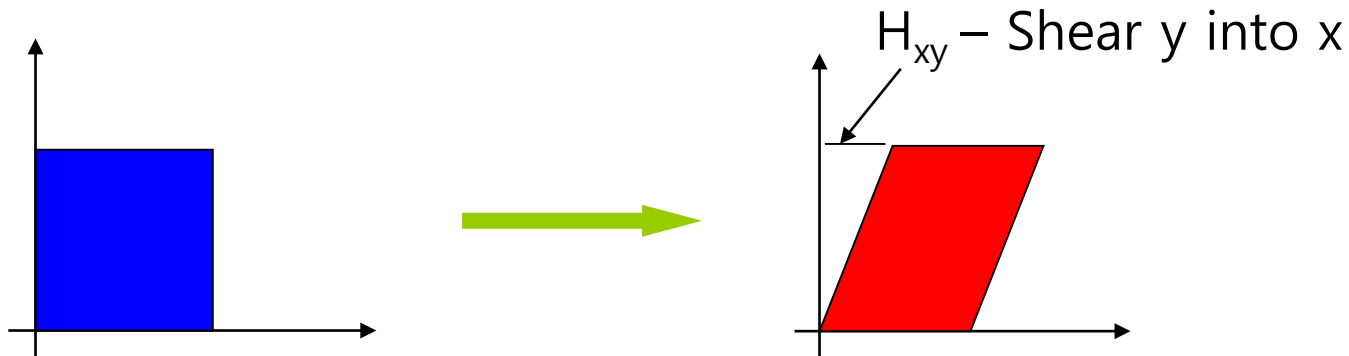
2D Shearing

- The Y-axis is not changed, and shearing applied in the X-axis direction:

$$x' = x + y \cdot h_{xy}$$

$$y' = y$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & h_{xy} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



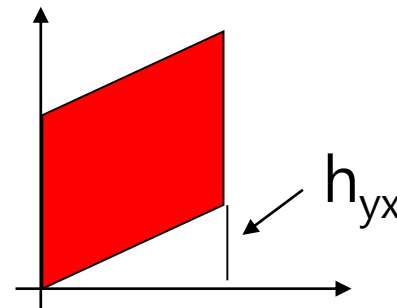
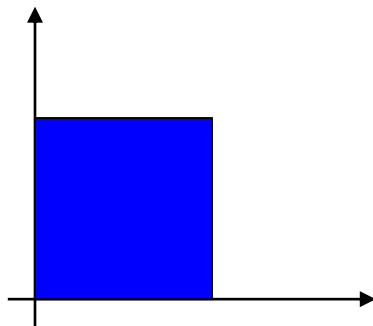
2D Shearing

- Shearing transformation does not change the size of object.
- The X-axis is not changed, and shearing applied in the Y-axis direction

$$x' = x$$

$$y' = x \cdot h_{yx} + y$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ h_{yx} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^* \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Homogeneous Coordinates

- ❑ In order to multiply translation, rotation, scaling transformation matrix, homogeneous coordinates are used.
- ❑ In homogeneous coordinates, the two-dimensional point $P(x, y)$ is expressed as $P(x, y, w)$.
- ❑ $(1, 2, 3)$ and $(2, 4, 6)$ represent the same homogeneous coordinates.
- ❑ If the w of the point $P(x, y, w)$ is 0, the point is located at an infinite point. If w is not 0, the point can be expressed as $(x/w, y/w, 1)$.

Transforming Homogeneous Coordinates□

$$T(dx, dy) = \begin{pmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{pmatrix}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S(sx, sy) = \begin{pmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The two-dimensional transformation matrix can be expressed as a 3x3 matrix of homogeneous coordinates.

3x3 2D Translation Matrix

□ Matrix-vector multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3x3 2D Rotation Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3x3 2D Scale Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3x3 2D Shearing Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & h_{xy} \\ h_{yx} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & h_{xy} & 0 \\ h_{yx} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Inverse 2D Transformation Matrix□

$$T^{-1} = \begin{pmatrix} 1 & 0 & -d_x \\ 0 & 1 & -d_y \\ 0 & 0 & 1 \end{pmatrix}$$

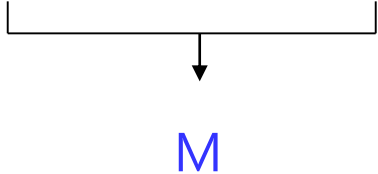
$$R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Composing Transformation

- ❑ *Composing transformation* is a process of forming one transformation by applying several transformation in sequence.
- ❑ If you want to transform one point, apply one transformation at a time or multiply the matrix and then multiply this matrix by the point.

$$Q = (M3 \cdot (M2 \cdot (M1 \cdot P))) = M3 \cdot M2 \cdot M1 \cdot P$$

(pre-multiply) 

M

- ❑ Matrix multiplication is associative.

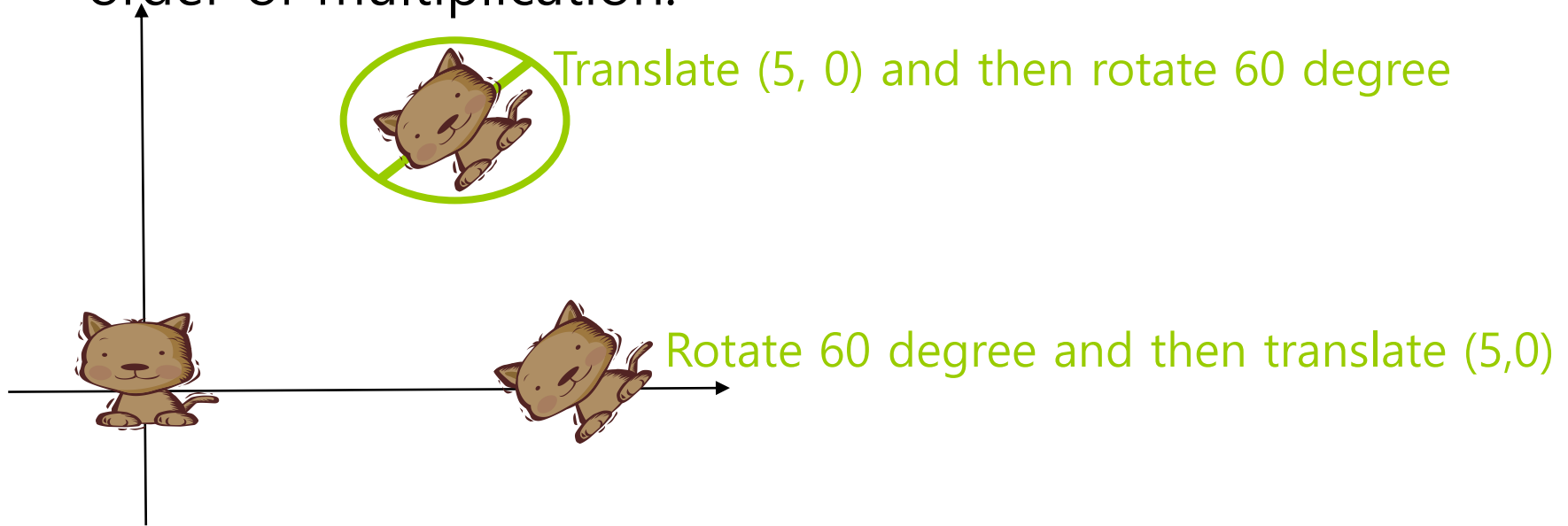
$$M3 \cdot M2 \cdot M1 = (M3 \cdot M2) \cdot M1 = M3 \cdot (M2 \cdot M1)$$

- ❑ Matrix multiplication is not commutative.

$$A \cdot B \neq B \cdot A$$

Transformation Order Matters!

- ❑ The multiplication of the transformation matrix is not commutative.
- ❑ Even if the transformation matrix is the same, it may have completely different results depending on the order of multiplication.

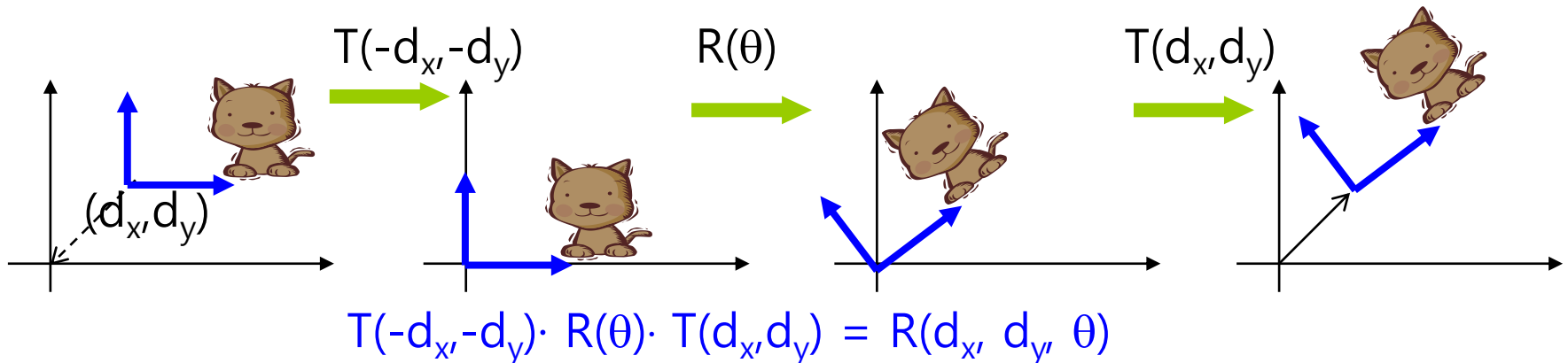


2D Rotate about an Arbitrary Pivot

■ Two-dimensional rotation by θ at an arbitrary pivot point $P(d_x, d_y)$:

1. $T(-d_x, -d_y)$
2. $R(\theta)$
3. $T(d_x, d_y)$

$$\begin{pmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -d_x \\ 0 & 1 & -d_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & d_x(1-\cos\theta)+d_y\sin\theta \\ \sin\theta & \cos\theta & d_y(1-\cos\theta)-d_x\sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

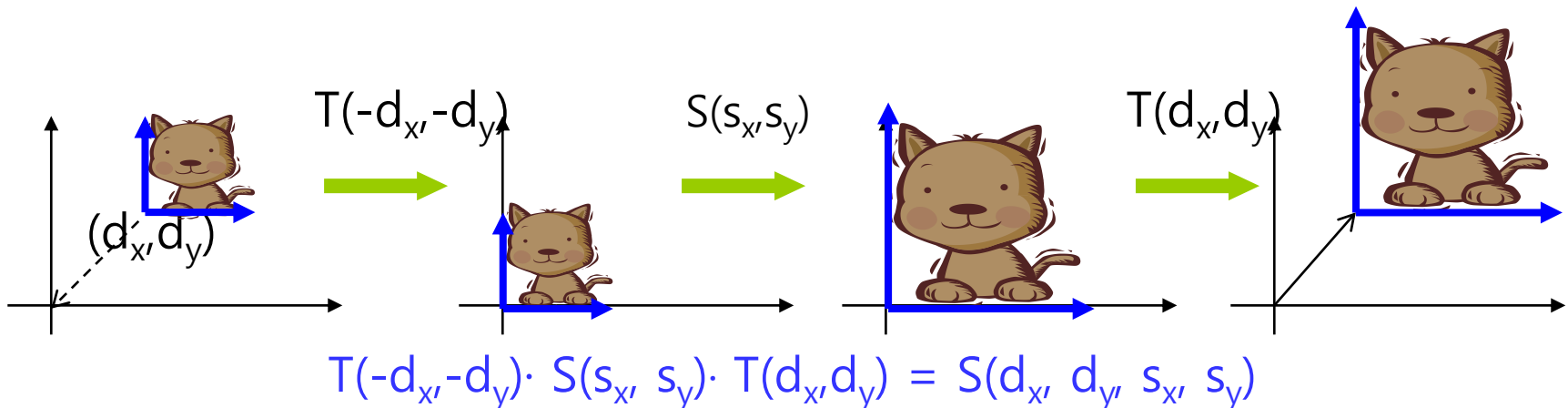


2D Scale about an Arbitrary Pivot

▣ Two-dimensional scaling an arbitrary pivot point $P(d_x, d_y)$:

1. $T(-d_x, -d_y)$
2. $S(s_x, s_y)$
3. $T(d_x, d_y)$

$$\begin{pmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -d_x \\ 0 & 1 & -d_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & d_x(1 - s_x) \\ 0 & s_y & d_y(1 - s_y) \\ 0 & 0 & 1 \end{pmatrix}$$

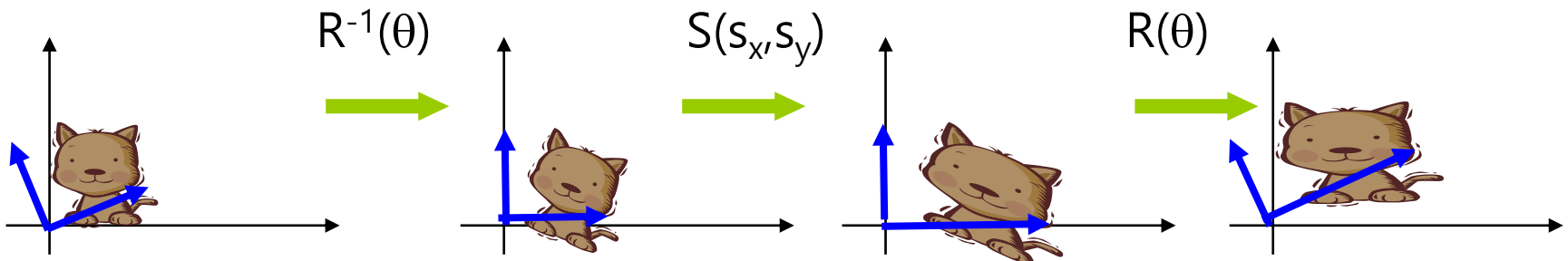


2D Scale in an Arbitrary Direction

- Two dimensional scaling in an arbitrary direction
(Rotating *the object to align the desired scaling directions with the coordinate axes* before scale transformation)

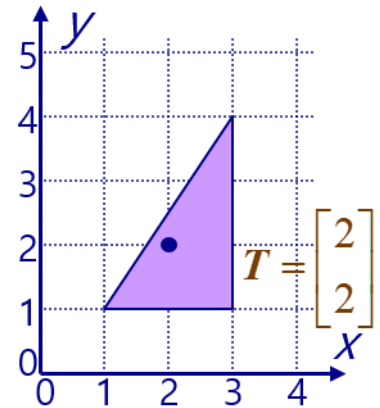
1. $R^{-1}(\theta)$
2. $S(s_x, s_y)$
3. $R(\theta)$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_x \cos^2\theta + s_y \sin^2\theta & (s_x - s_y) \cos\theta \sin\theta & 0 \\ (s_x - s_y) \cos\theta \sin\theta & s_y \cos^2\theta + s_x \sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Example: 2D Rotate about an Arbitrary Pivot

Rotate a triangle with vertices (1,1), (3,1), (3,4) by 45 degrees about the pivot point (2,2).



1. Translate point to origin

$$\mathbf{T}_1 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

2. Rotate 45 degrees

$$\mathbf{R} = \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix}$$

3. Translate back to original location $\mathbf{T}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

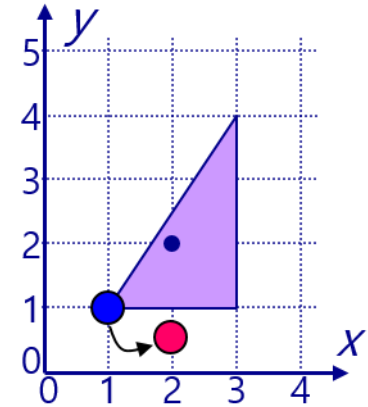
4. Composite transformation $\mathbf{P}' = \mathbf{R}(\mathbf{P} + \mathbf{T}_1) + \mathbf{T}_2$

$$\mathbf{P}' = \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Example: 2D Rotate about an Arbitrary Pivot

□ $P_1 (1, 1)$

$$\begin{aligned}P_1' &= \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\&= \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\&= \begin{bmatrix} 0 \\ -1.414 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\&= \begin{bmatrix} 2 \\ 0.586 \end{bmatrix}\end{aligned}$$



Example: 2D Rotate about an Arbitrary Pivot

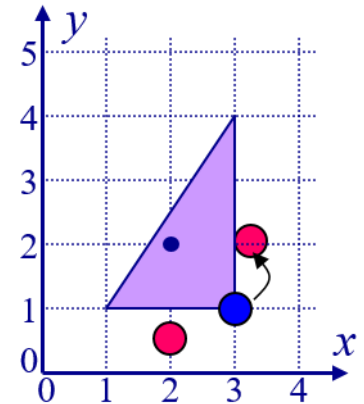
□ $P_2 (3, 1)$

$$P_2' = \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.414 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3.414 \\ 2 \end{bmatrix}$$



Example: 2D Rotate about an Arbitrary Pivot

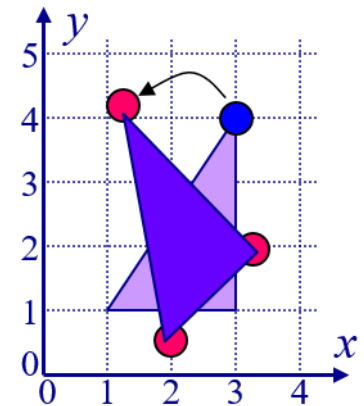
□ $P_3 (3, 4)$

$$P_3' = \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

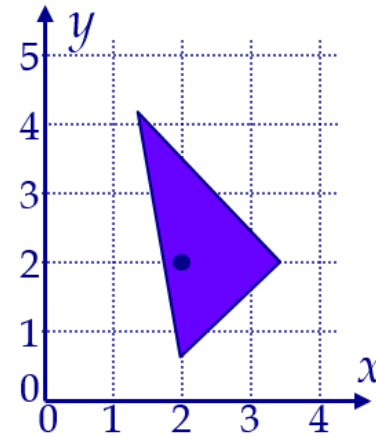
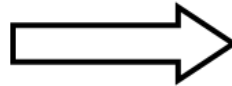
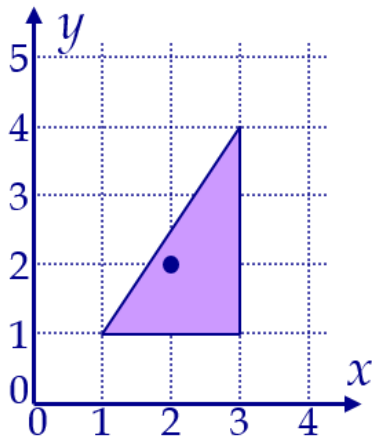
$$= \begin{bmatrix} -.707 \\ 2.121 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.293 \\ 4.121 \end{bmatrix}$$



Example: 2D Rotate about an Arbitrary Pivot

□ Result:



Before:

$(1, 1), (3, 1), (3, 4)$

After:

$(2, 0.59), (3.41, 2), (1.29, 4.2)$

Example: 2D Rotate about an Arbitrary Pivot Using Composite Transformation Matrix

- Rotate a triangle with vertices (1,1), (3,1), (3,4) by 45 degrees about the pivot point (2,2).
- $P' = T(2,2)R(45)T(-2,-2)P = M P$

$$M = T_{(2,2)}R_{45}T_{(-2,-2)}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) & 0 \\ \sin(45^\circ) & \cos(45^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & -.828 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{M}$$

Example: 2D Rotate about an Arbitrary Pivot Using Composite Transformation Matrix

1. P_1

$$P_1' = MP_1 = \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & -.828 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ .586 \\ 1 \end{bmatrix}$$

2. P_2

$$P_2' = MP_2 = \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & -.828 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.414 \\ 2 \\ 1 \end{bmatrix}$$

3. P_3

$$P_3' = MP_3 = \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & -.828 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.293 \\ 4.121 \\ 1 \end{bmatrix}$$

