

Transformation

Fall 2025

10/9/2025

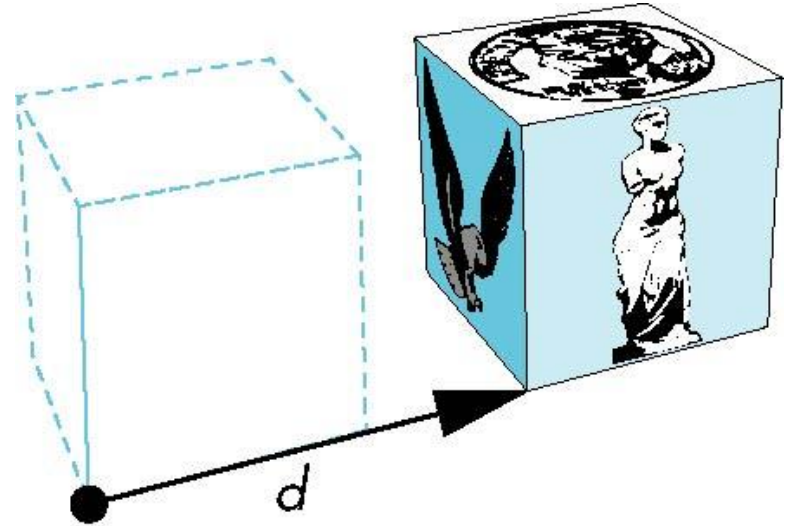
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3D Transformations

- ❑ In general, three-dimensional transformation can be thought of as an extension of two-dimensional transformation.
- ❑ The basic principles of three-dimensional translation, scaling, shearing are the same as those of two-dimensional.
- ❑ However, three-dimensional rotation is a bit more complicated.

3D Translation

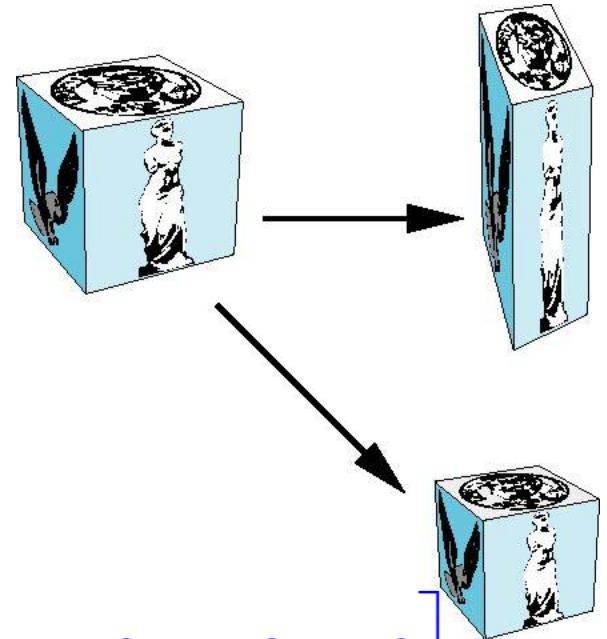
$$p' = p + d \quad p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \quad d = \begin{bmatrix} dx \\ dy \\ dz \\ 0 \end{bmatrix}$$



$$p' = Tp \quad T = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -dx \\ 0 & 1 & 0 & -dy \\ 0 & 0 & 1 & -dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Scale

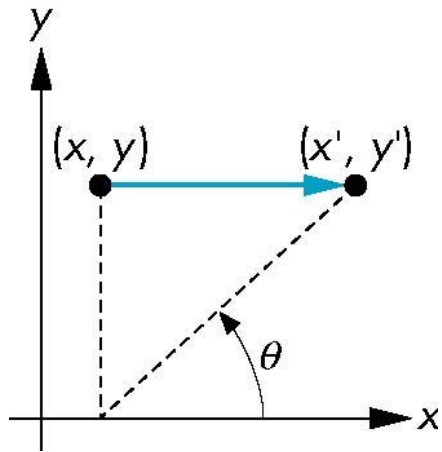
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$p' = Sp \quad S = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} \frac{1}{sx} & 0 & 0 & 0 \\ 0 & \frac{1}{sy} & 0 & 0 \\ 0 & 0 & \frac{1}{sz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

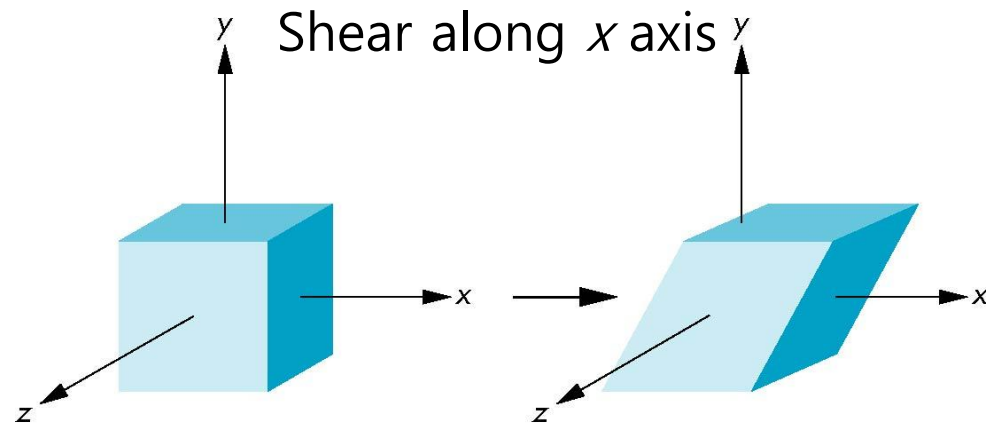
3D Shear



$$x' = x + y \cot \theta$$

$$y' = y$$

$$z' = z$$



$$\mathbf{H}_{xy}(\theta) = \begin{bmatrix} 1 & \mathbf{\cot \theta} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tan \theta = \frac{y}{x' - x} \Rightarrow \cot \theta = \frac{x' - x}{y}$$

3D Rotation

$$R^{-1}(\theta) = R(-\theta)$$

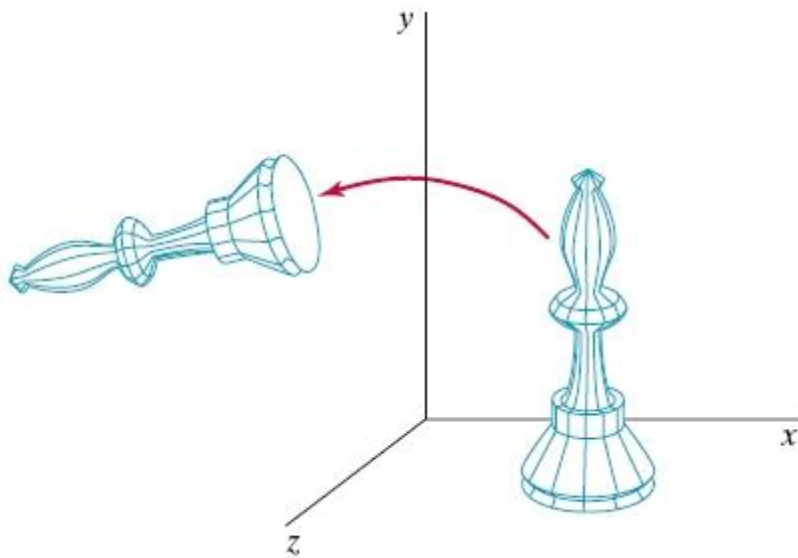
$$R^{-1}(\theta) = R^T(\theta)$$

3D rotation in Z-axis

$$x' = x \cos\theta - y \sin\theta$$

$$y' = x \sin\theta + y \cos\theta$$

$$z' = z$$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_Z(\theta) \cdot P$$

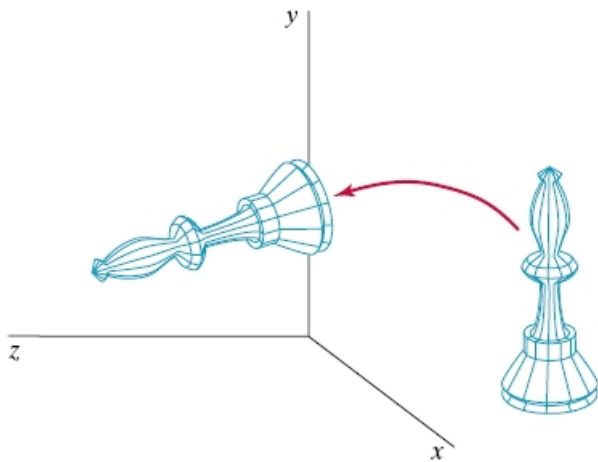
3D Rotation

3D rotation in X-axis

$$y' = y \cos\theta - z \sin\theta$$

$$z' = y \sin\theta + z \cos\theta$$

$$x' = x$$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_X(\theta) \cdot P$$

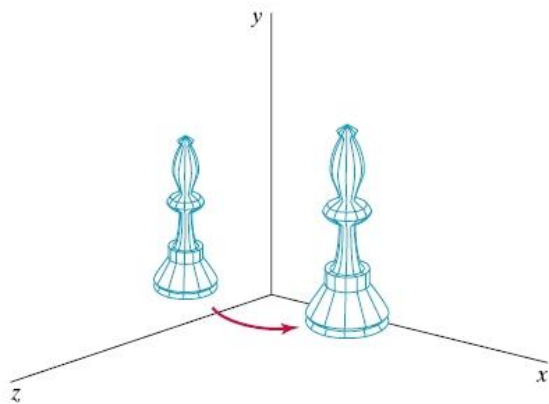
3D Rotation

▣ 3D rotation in Y-axis

$$x' = x \cos\theta + z \sin\theta$$

$$z' = -x \sin\theta + z \cos\theta$$

$$y' = y$$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

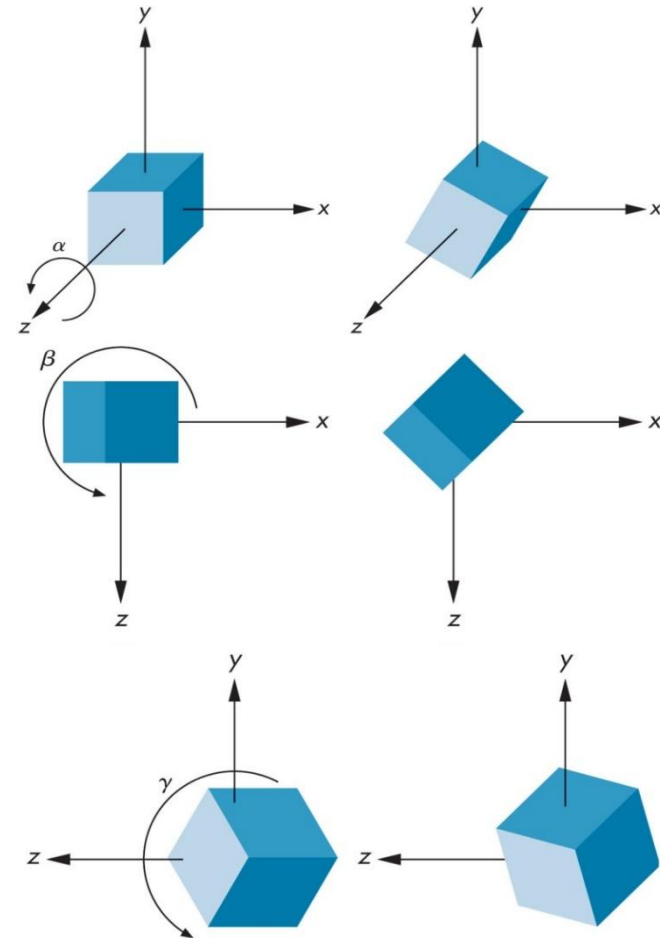
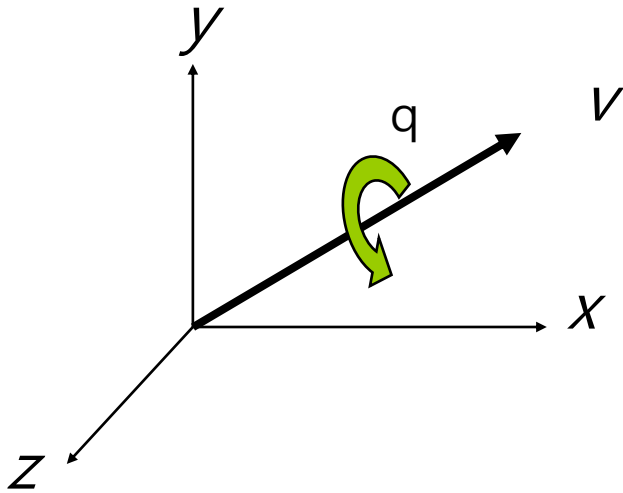
$$P' = R_Y(\theta) \cdot P$$

3D Rotation about the Origin

- A rotation by θ about an arbitrary axis can be decomposed into the concatenation of rotations about the x , y , and z axes.

$$\mathbf{R}(\theta) = \mathbf{R}_Z(\theta_Z)\mathbf{R}_Y(\theta_Y)\mathbf{R}_X(\theta_X)$$

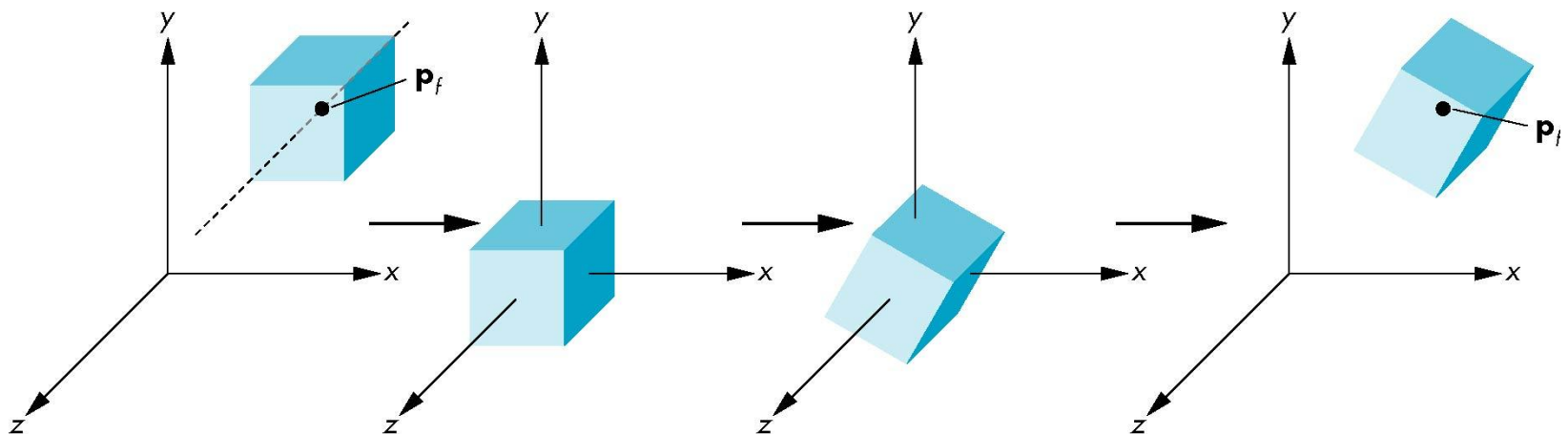
θ_X , θ_Y , θ_Z are called the Euler angles.



Rotation About a Pivot other than the Origin

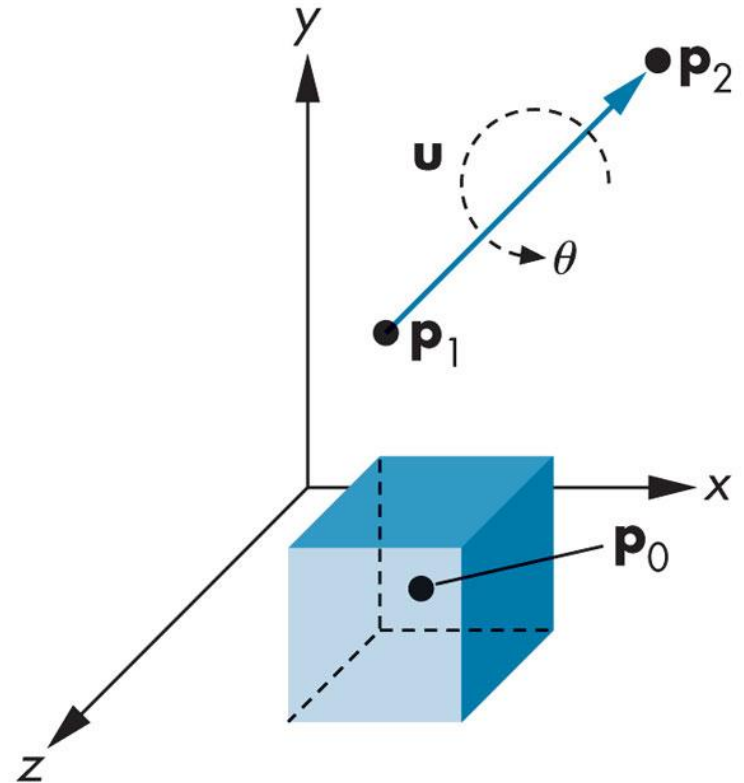
- Move fixed point to origin, rotate, and then move fixed point back.
- $\mathbf{M} = \mathbf{T}(\mathbf{p}_f) \mathbf{R}_Z(\theta) \mathbf{T}(-\mathbf{p}_f)$

$$\mathbf{M} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & x_f - x_f \cos \theta + y_f \sin \theta \\ \sin \theta & \cos \theta & 0 & y_f - x_f \sin \theta - y_f \cos \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3D Rotation about an Arbitrary Axis

- Move P_1 to the origin.
- Rotate twice to align the arbitrary axis u with the Z-axis.
- Rotate by θ in Z-axis.
- Undo two rotations (undo alignment).
- Move back to P_1 .



$$M = T(P_1) R_x(-\theta_x) R_y(-\theta_y) R_z(\theta) R_y(\theta_y) R_x(\theta_x) T(-P_1)$$

3D Rotation about an Arbitrary Axis

- ▣ The translation matrix, $T(-P_1)$

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

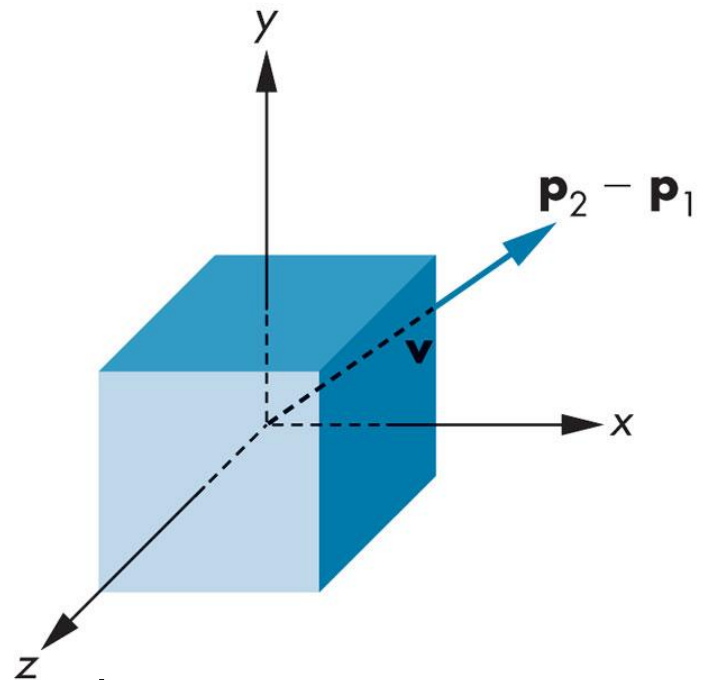
3D Rotation about an Arbitrary Axis

- The rotation-axis vector

$$\begin{aligned} u &= P_2 - P_1 \\ &= (x_2 - x_1, y_2 - y_1, z_2 - z_1) \end{aligned}$$

- Normalize u :

$$v = \frac{u}{\|u\|} = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$



- Rotate along x -axis until v hits xz -plane
- Rotate along y -axis until v hits z -axis

3D Rotation about an Arbitrary Axis

- Find θ_x and θ_y

$$\mathbf{v} = (\alpha_x, \alpha_y, \alpha_z)$$

$$\alpha_x^2 + \alpha_y^2 + \alpha_z^2 = 1$$

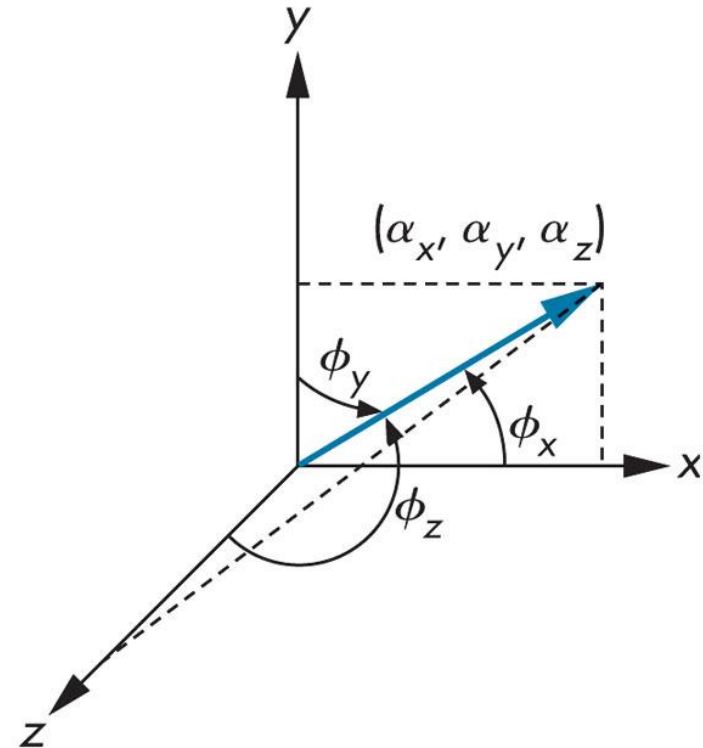
- Direction cosines:

$$\cos \phi_x = \alpha_x$$

$$\cos \phi_y = \alpha_y$$

$$\cos \phi_z = \alpha_z$$

$$\cos^2 \phi_x + \cos^2 \phi_y + \cos^2 \phi_z = 1$$

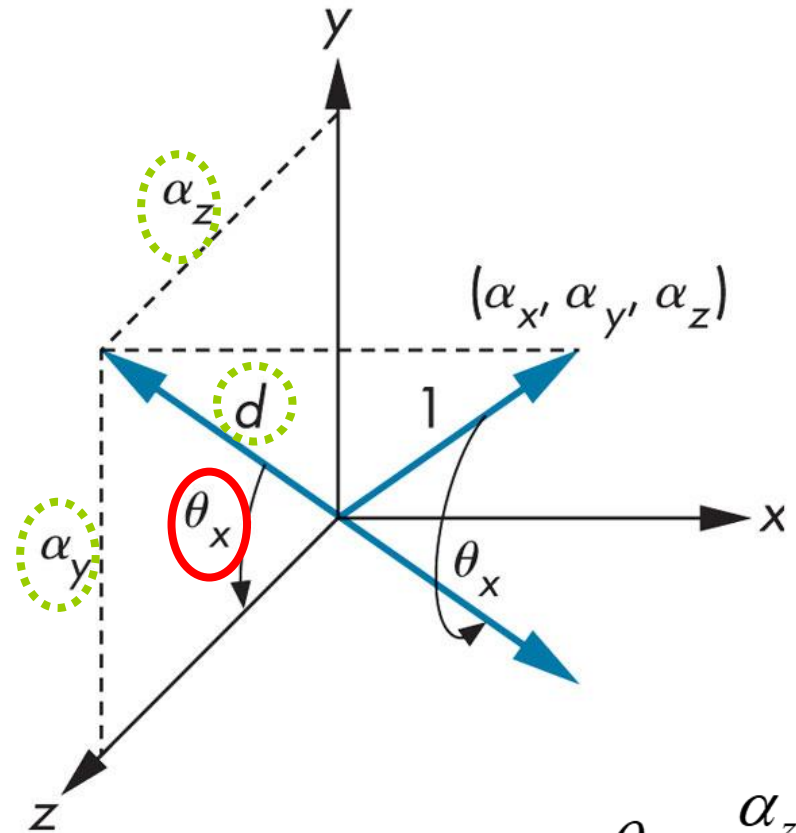


3D Rotation about an Arbitrary Axis

- Compute x-rotation θ_x

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha_z}{d} & -\frac{\alpha_y}{d} & 0 \\ 0 & \frac{\alpha_y}{d} & \frac{\alpha_z}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$



$$\cos \theta_x = \frac{\alpha_z}{d}$$

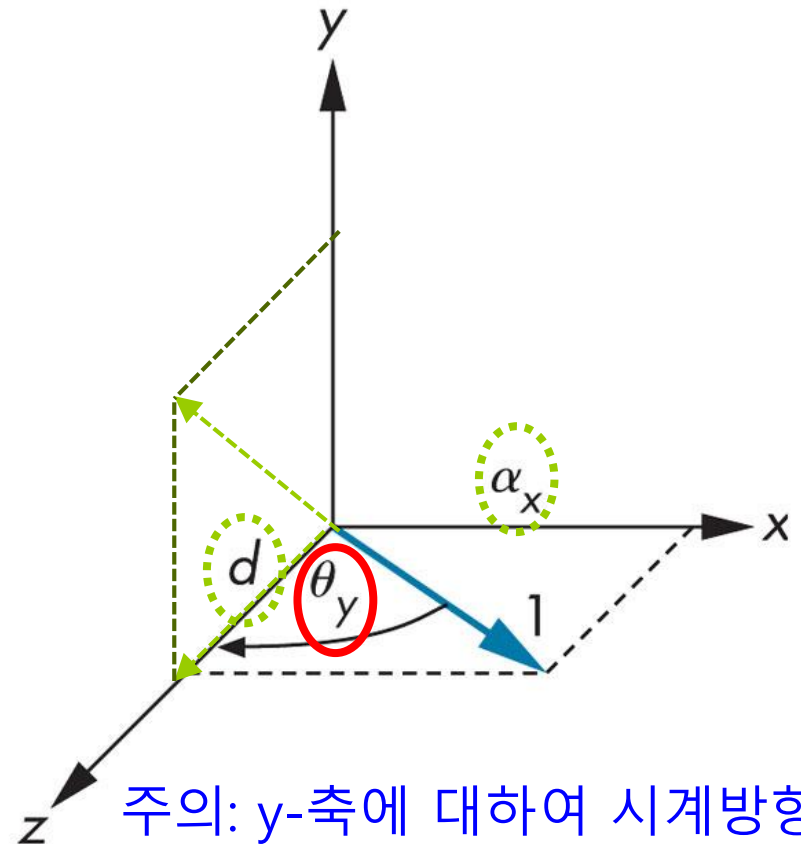
$$\sin \theta_x = \frac{\alpha_y}{d}$$

3D Rotation about an Arbitrary Axis

- Compute y-rotation θ_y

$$R_y(\theta_y) = \begin{bmatrix} d & 0 & -\alpha_x & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$



$$\cos \theta_y = d$$

$$\sin \theta_y = \alpha_x$$

3D Rotation about an Arbitrary Axis

- Rotation about the z axis

$$R_z(\theta_z) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

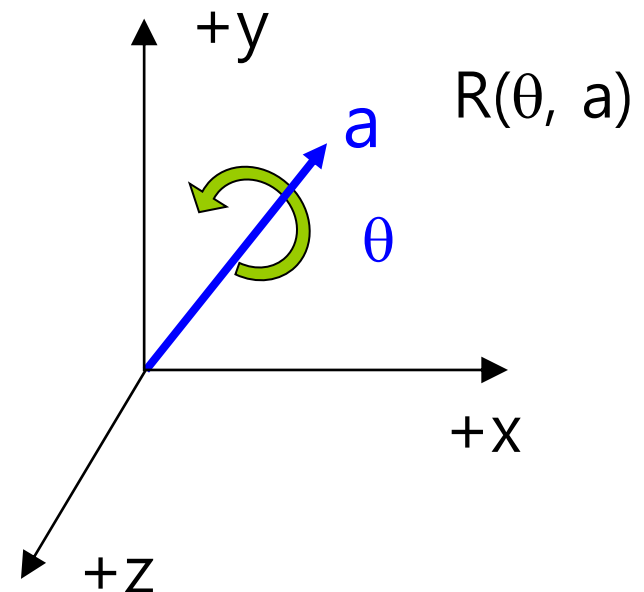
- Undo alignment, $R_x(-\theta_x)R_y(-\theta_y)$

- Undo translation, $T(P_1)$

- $M = T(P_1)R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x)T(-P_1)$

3D Rotation about an Arbitrary Axis Using Rotation Vectors

- 3D rotation can be expressed as 4 numbers of one angle of rotation about an arbitrary axis (a_x, a_y, a_z).
- It consists of a unit vector a (x, y, z) representing an arbitrary axis of rotation and a value of θ ($0 \sim 360$ degrees) representing the rotation angle around the unit vector.
- 3D rotation vector



3D Rotation about an Arbitrary Axis

▣ From axis/angle, we make the following rotation matrix.

$$R = I \cos \theta + \text{Symmetric} (1 - \cos \theta) + \text{Skew} \sin \theta$$

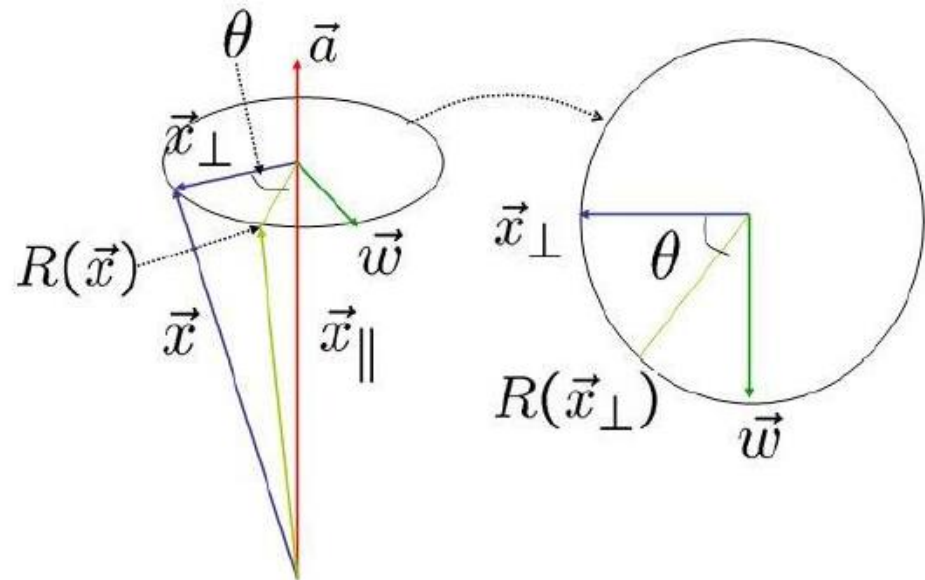
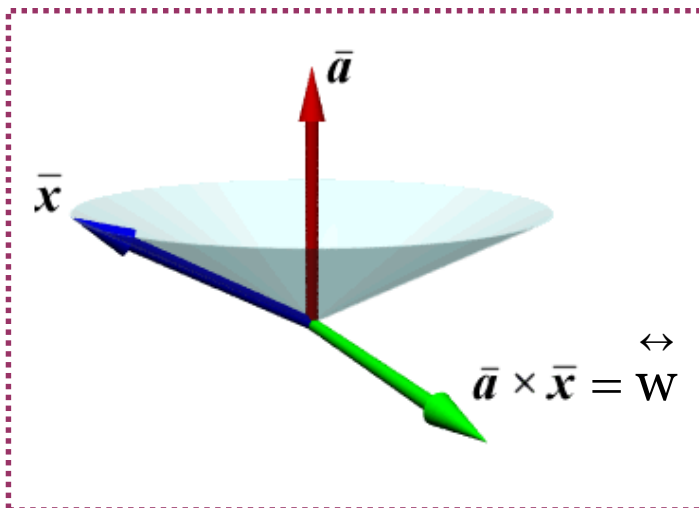
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cos \theta + \begin{bmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{bmatrix} (1 - \cos \theta) + \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \sin \theta$$

$$= \begin{bmatrix} a_x^2 + \cos \theta (1 - a_x^2) & a_x a_y (1 - \cos \theta) - a_z \sin \theta & a_x a_z (1 - \cos \theta) + a_y \sin \theta \\ a_x a_y (1 - \cos \theta) + a_z \sin \theta & a_y^2 + \cos \theta (1 - a_y^2) & a_y a_z (1 - \cos \theta) - a_x \sin \theta \\ a_x a_z (1 - \cos \theta) - a_y \sin \theta & a_y a_z (1 - \cos \theta) + a_x \sin \theta & a_z^2 + \cos \theta (1 - a_z^2) \end{bmatrix}$$

3D Rotation as Vector Components

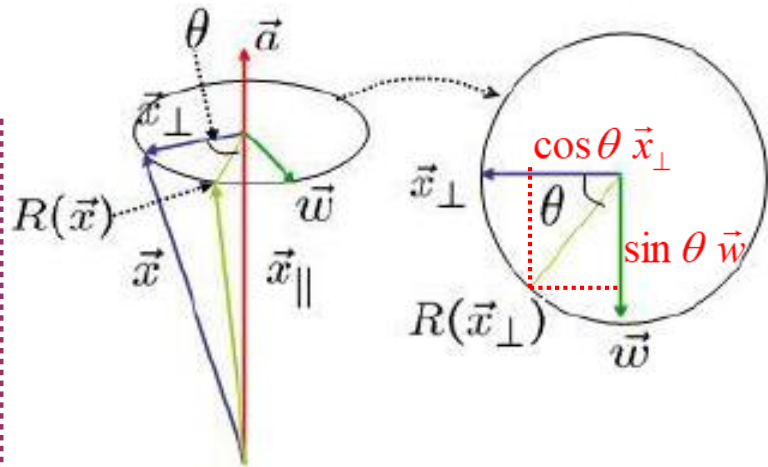
- 3D rotation by θ around the arbitrary axis $\mathbf{a} = [a_x, a_y, a_z]$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \left(\text{Symmetric} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} (1 - \cos \theta) + \text{Skew} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \sin \theta + \mathbf{I} \cos \theta \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



3D Rotation as Vector Components

$$\begin{aligned}
 \vec{w} &= \vec{a} \times \vec{x}_{\perp} \\
 &= \vec{a} \times (\vec{x} - \vec{x}_{\parallel}) \\
 &= (\vec{a} \times \vec{x}) - (\vec{a} \times \vec{x}_{\parallel}) \\
 &= \vec{a} \times \vec{x}
 \end{aligned}$$



$$R(\vec{x}_{\perp}) = \cos \theta \vec{x}_{\perp} + \sin \theta \vec{w}$$

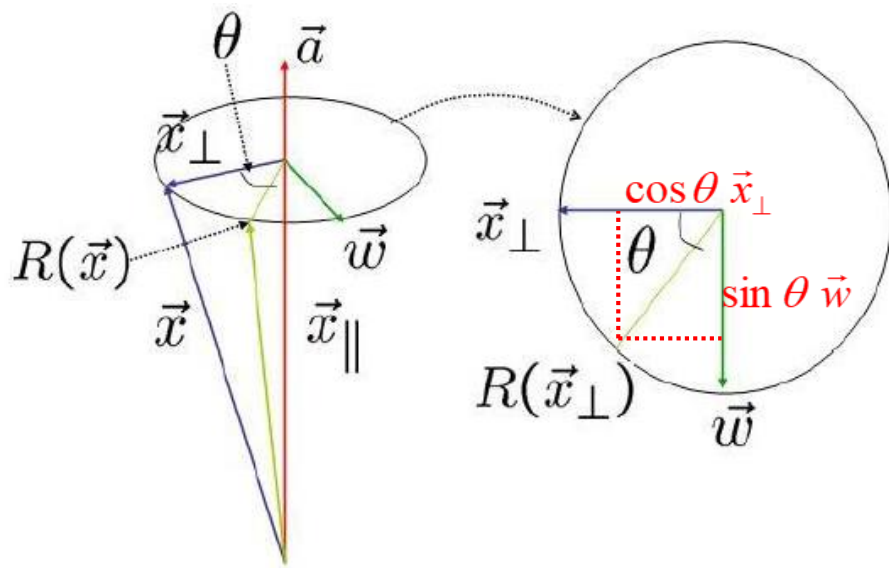
$$\begin{aligned}
 \vec{x}_{\parallel} &= (\vec{a} \cdot \vec{x}) \vec{a} \\
 \vec{x}_{\perp} &= \vec{x} - \vec{x}_{\parallel} = \vec{x} - (\vec{a} \cdot \vec{x}) \vec{a}
 \end{aligned}$$

$$\begin{aligned}
 R(\vec{x}) &= R(\vec{x}_{\parallel}) + R(\vec{x}_{\perp}) \\
 &= R(\vec{x}_{\parallel}) + \cos \theta \vec{x}_{\perp} + \sin \theta \vec{w} \\
 &= (\vec{a} \cdot \vec{x}) \vec{a} + \cos \theta (\vec{x} - (\vec{a} \cdot \vec{x}) \vec{a}) + \sin \theta \vec{w} \\
 &= \cos \theta \vec{x} + (1 - \cos \theta) (\vec{a} \cdot \vec{x}) \vec{a} + \sin \theta (\vec{a} \times \vec{x})
 \end{aligned}$$

Symmetric

Skew

3D Rotation as Vector Components



$$\begin{aligned}
 R(x_{\perp}) &= \|R(x_{\perp})\| \cos \theta x_{\perp} + \|R(x_{\perp})\| \sin \theta w \\
 &= \|x_{\perp}\| \cos \theta x_{\perp} + \|x_{\perp}\| \sin \theta w \\
 &= \|x_{\perp}\| \cos \theta \frac{x_{\perp}}{\|x_{\perp}\|} + \|x_{\perp}\| \sin \theta \frac{w}{\|w\|} \\
 &= \|x_{\perp}\| \cos \theta \frac{x_{\perp}}{\|x_{\perp}\|} + \|x_{\perp}\| \sin \theta \frac{w}{\|x_{\perp}\|} \\
 &= \cos \theta x_{\perp} + \sin \theta w
 \end{aligned}$$

3D Rotation as Vector Components

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \left(\mathbf{Symmetric} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} (1 - \cos \theta) + \mathbf{Skew} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \sin \theta + \mathbf{I} \cos \theta \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- The vector a specifies the axis of rotation. This axis vector must be normalized.
- The rotation angle is given by θ .
- The basic idea is that *any rotation can be decomposed into weighted contributions from three different vectors*.

3D Rotation as Vector Components

- *The symmetric matrix of a vector* generates a vector in the direction of the axis.
- The symmetric matrix is composed of the outer product of a row vector and an column vector of the same value.

$$\text{Symmetric} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} = \begin{bmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{bmatrix}$$

$$\text{Symmetric} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \bar{a}(\bar{a} \cdot \bar{x})$$

3D Rotation as Vector Components

- ▣ *Skew symmetric matrix of a vector* generates a vector that is perpendicular to both the axis and it's input vector.

$$\text{Skew}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\text{Skew}(\bar{a})\bar{x} = \bar{a} \times \bar{x}$$

3D Rotation as Vector Components

- First, consider a rotation by 0. :

$$\text{Rotate} \left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, 0 \right) = \begin{bmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{bmatrix} (1-1) + \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} 0 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} 1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- For instance, a rotation about the x-axis:

$$\text{Rotate} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \theta \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (1 - \cos \theta) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \sin \theta + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cos \theta$$

$$\text{Rotate} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \theta \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

3D Rotation as Vector Components

- For instance, a rotation about the y-axis:

$$\text{Rotate} \begin{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \theta \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} (1 - \cos \theta) + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \sin \theta + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cos \theta$$
$$\text{Rotate} \begin{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \theta \end{pmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

- For instance, a rotation about the z-axis:

$$\text{Rotate} \begin{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (1 - \cos \theta) + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sin \theta + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cos \theta$$
$$\text{Rotate} \begin{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta \end{pmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$