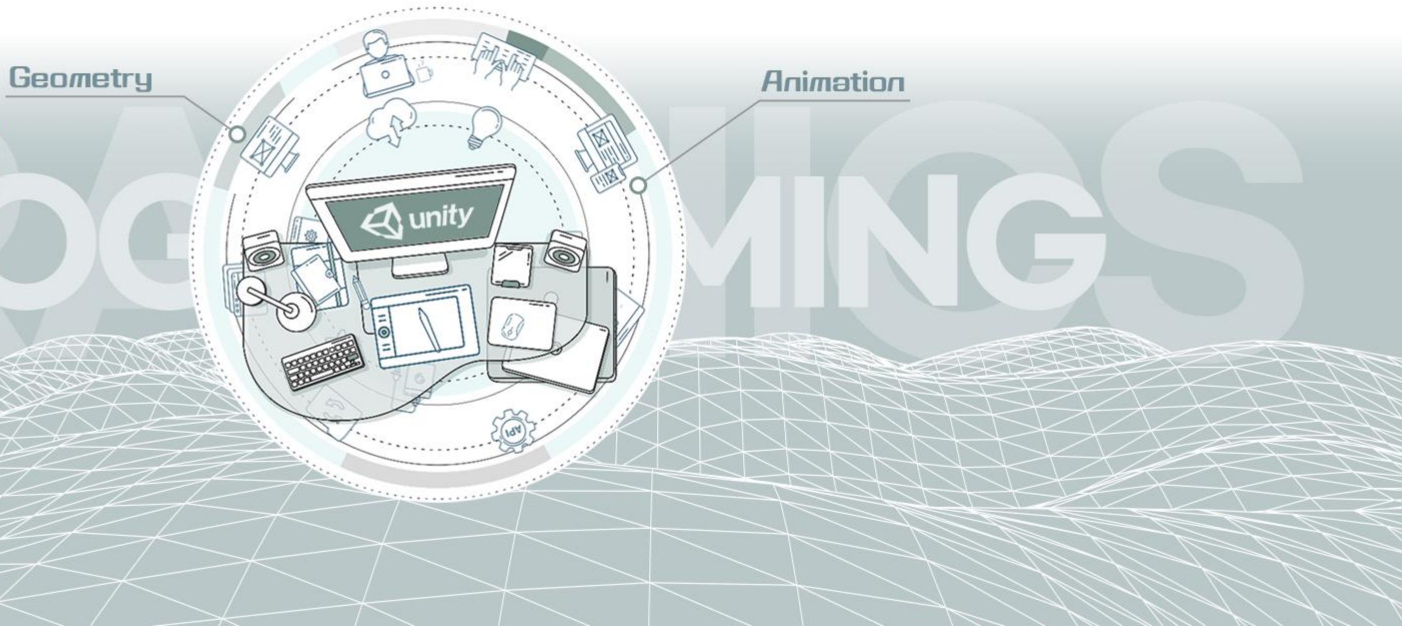
유니티(Unity)를 활용한

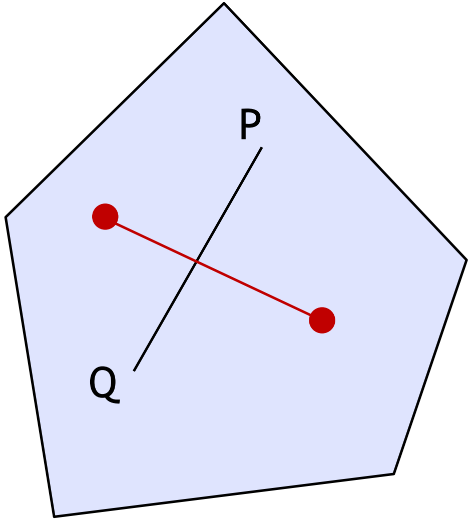
그래픽스 프로그래밍

04 Geometric Objects-Spaces and Matrix(2)

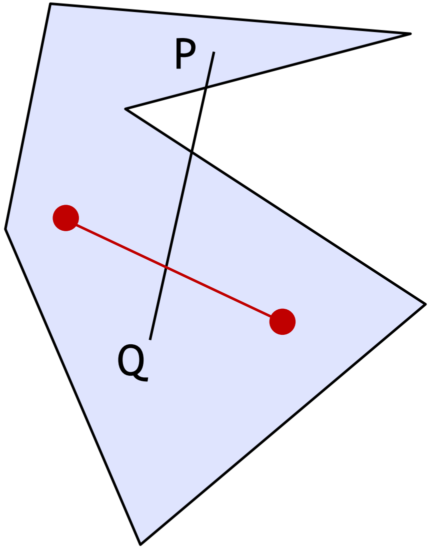


Convexity

- An object is convex if only if for any two points in the object all points on the line segment between these points are also in the object.



convex



not convex
(=concave)

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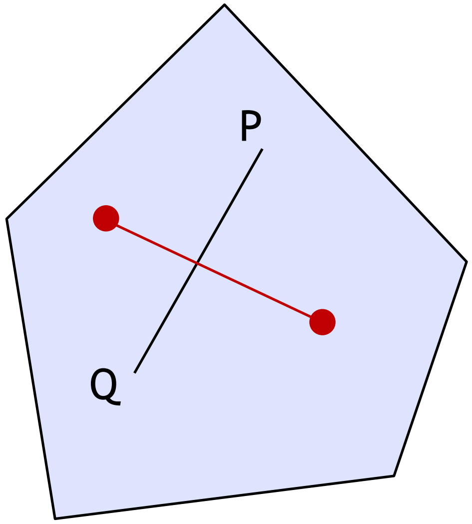
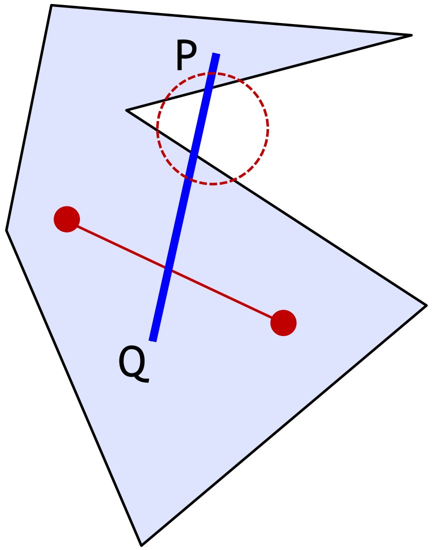


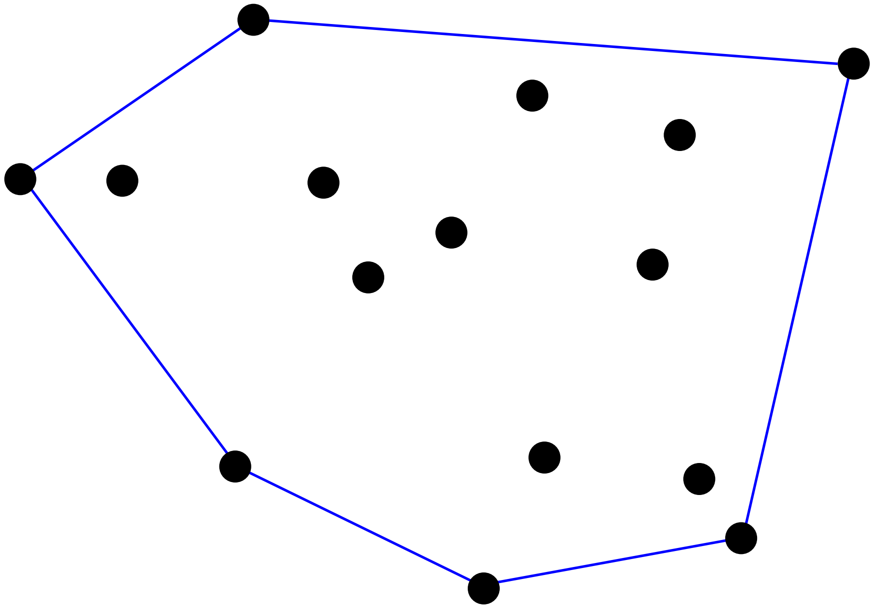
convex



not convex
(=concave)

Convex Hull

- Smallest convex object containing $P_1,P_2 \ldots P_n$
- Formed by “shrink wrapping” points



Affine Sums

- The affine sum of the points defined by $P_1, P_2 \ldots P_n$ is

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_n P_n$$

Can show by induction that this sum makes sense iff

$$\alpha_1 + \alpha_2 + \cdots \alpha_n = 1$$

$P = \alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_n P_n$

If, in addition, $\alpha_i > 0, \ i = 1, 2, \ldots n,$ we have the convex hull of $P_1, P_2 \ldots P_n$.

$P = \alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_n P_n$

Convex hull $\{P_1, P_2 \ldots P_n\}$, you can see that it includes all the line segments connecting the pairs of points.

Linear/Affine Combination of Vectors

- Linear combination of m vectors
  - Vector $v_1, v_2 \dots v_m$
  - $\mathrm{w} = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$ *where* $\alpha_1, \alpha_2 \dots \alpha_m$ *are scalars*
- If the sum of the scalar values, $\alpha_1, \alpha_2 \dots \alpha_m$ is 1, it becomes an affine combination.
  - $\alpha_1 + \alpha_2 + \cdots \alpha_m = 1$

Convex Combination

- If, in addition, $\alpha_i > 0, \; i = 1, 2, \dots n,$ we have the convex hull of $P_1, P_2 \dots P_n$.
- Therefore, the linear combination of vectors satisfying the following condition is a convex.

$$\alpha_1 + \alpha_2 + \cdots \alpha_m = 1$$

$$and$$

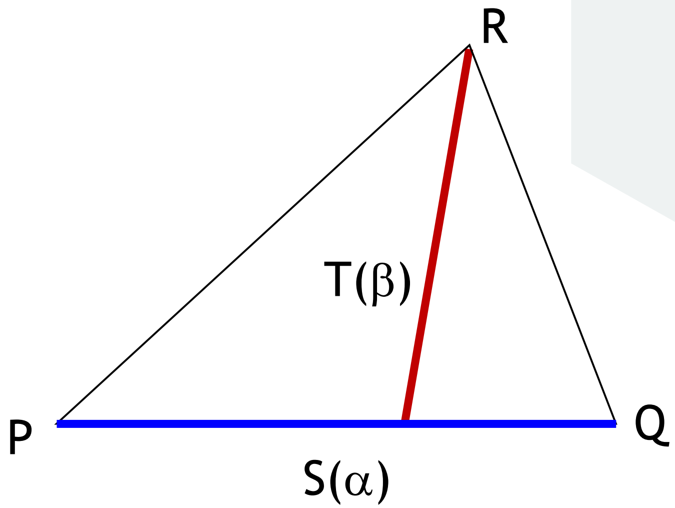
$$\alpha_i \geq 0 \; for \; i = 1, 2 \dots m$$

$$\alpha_i \; is \; between \; 0 \; and \; 1$$

- Convexity
  - Convex hull

Plane

- A plane can be defined by a point and two vectors or by three points.
- Suppose 3 points, P, Q, R
- Line segment PQ
  - $S(\alpha) = \alpha P + (1 - \alpha)Q$
- Line segment SR
  - $T(\beta) = \beta S + (1 - \beta)R$



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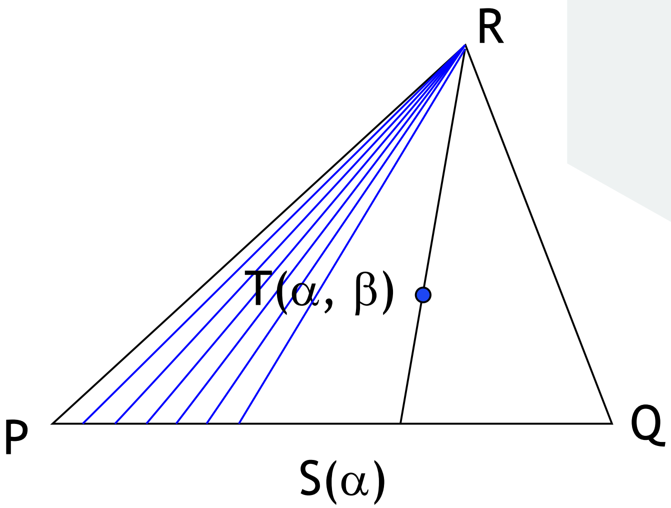
  - $T(\beta) = \beta S + (1-\beta)R$
- Plane defined by P, Q, R
  - $T(\alpha, \beta) = \beta(\alpha P + (1-\alpha)Q) + (1-\beta)R$

    $= P + \beta(1-\alpha)(Q-P) + (1-\beta)(R-P)$
  - $for\ 0 \le \alpha, \beta \le 1, we\ get\ all\ points\ in\ triangle, T(\alpha, \beta)$



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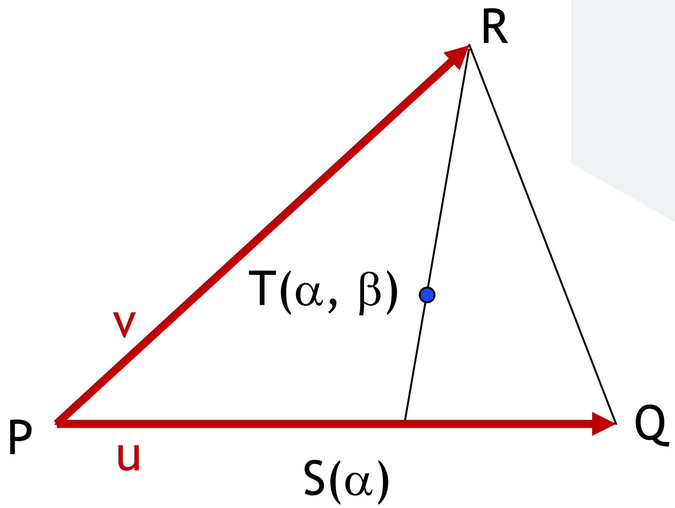
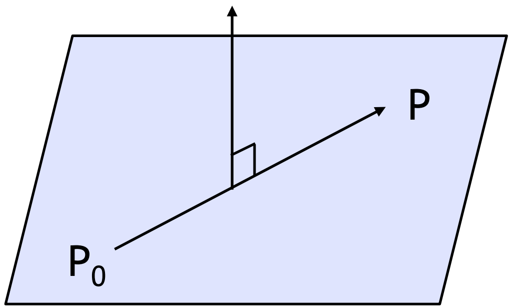
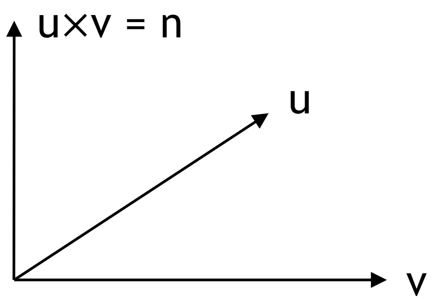
    $= P + \beta(1-\alpha)(Q-P) + (1-\beta)(R-P) = P + \beta(1-\alpha)u + (1-\beta)v$
  - $for\ 0 \leq \alpha,\ \beta \leq 1, we\ get\ all\ points\ in\ triangle, T(\alpha,\ \beta)$



Plane

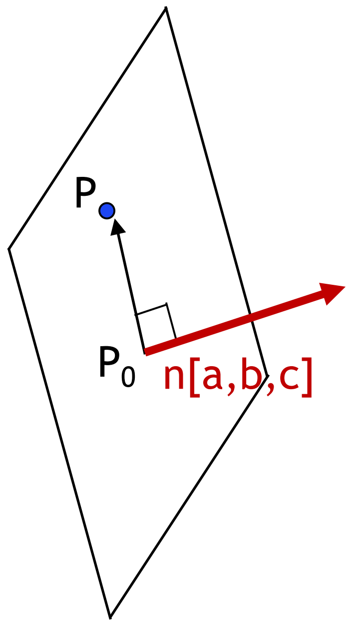
- Plane equation defined by a point P_0 and two non parallel vectors, u, v
  - $T(\alpha,\ \beta) = P_0 + \alpha u + +\beta v$
  - $P - P_0 = \alpha u + +\beta v$ (P is point on the plane)
- Using n (the cross product of u, v), the plane equation is as follows
  - $n \cdot (P - P_0) = 0\ (where\ n = u \times v\ and\ n\ is\ a\ normal\ vector)$





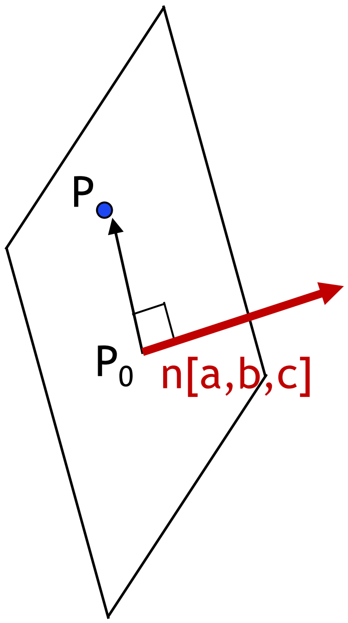
Plane

- The plane is represented by a normal vector n and a point P_0 on the plane.
  - $Plane\ (n, d)\ where\ n\ (a, b, c)$
  - $ax + by + cz + d = 0$
  - $n \cdot P + d = 0$
  - $d = n \cdot P$
- For point p on the plane, n•(p - p_0) =0
- If the plane normal n is a unit vector,
  then n•p + d gives the shortest signed distance from
  the plane to point p : d = -n•p



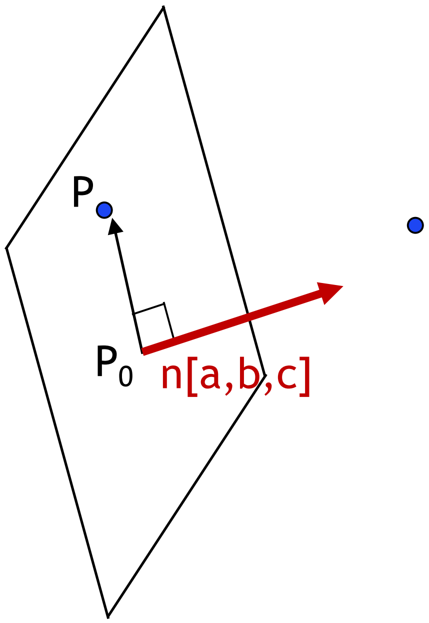
Relationship between Point and Plane

- Relationship between point p and plane (n, d)
  - $If\ n \cdot P + d = 0$, then p is in the plane.



Relationship between Point and Plane

- Relationship between point p and plane (n, d)
  - $If\ n \cdot P + d = 0$, then p is in the plane.
  - $If\ n \cdot P + d > 0$, then p is outside the plane.



Relationship between Point and Plane

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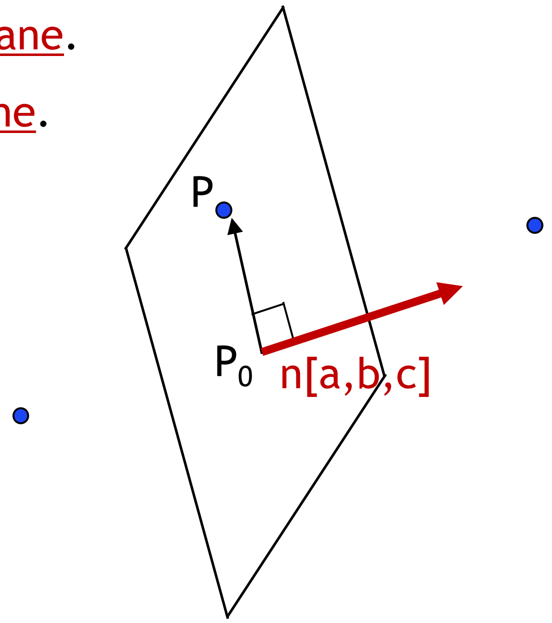
  - $If\ n \cdot P + d < 0$, then p is inside the plane.



Plane Normalization

- Plane normalization
  - Normalize the plane normal vector
  - Since the length of the normal vector affects the constant d, d is also normalized.

$$\frac{1}{\|n\|}(n, d) = \left(\frac{n}{\|n\|}, \frac{d}{\|n\|}\right)$$

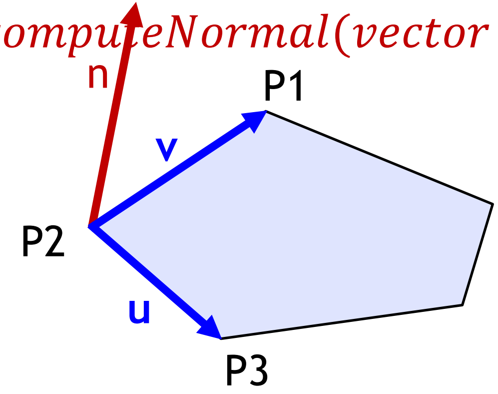
Computing a Normal from 3 Points in Plane

- Find the normal from the polygon's vertices.
  - The polygon's normal computes two non-collinear edges. (assuming that no two adjacent edges will be collinear)
  - Then, normalize it after the cross product.

$\} \quad return\ n.normalize(); \quad else \quad return\ y;$

$== 0) \quad n = cross(u, v); \quad v = P1 - P2; \quad u$

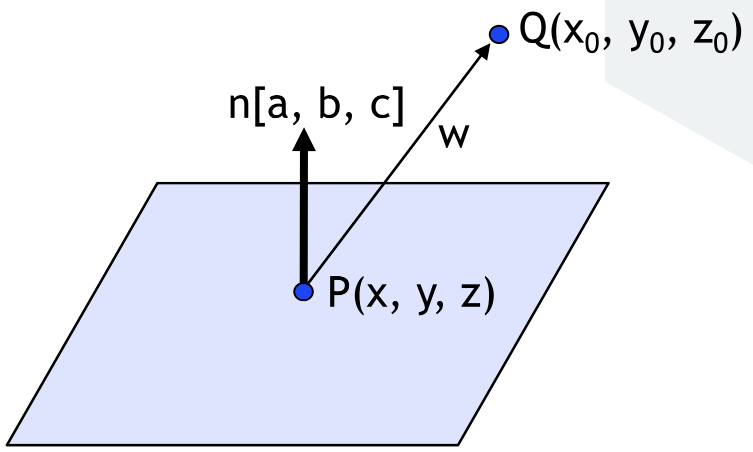
$= P3 - P2; \quad vector\ u, v, n, y(0, 1, 0); \ void\ computeNormal(vector\ P1, vector$



Computing a Distance from Point to Plane

- Find the closest distance to a plane (n, d) in space and a point Q out of the plane.
  - The plane's normal is n, and D is the distance between a point P and a point Q on the plane.

$$\mathrm{w} = Q - P = [x_0 - x, y_0 - y, z_0 - z]$$



$$Projecting\ w\ onto\ n : w = n\frac{\|w \cdot n\|}{\|n^2\|}\ \&\ \|w\| = \frac{|w \cdot n|}{\|n\|}$$

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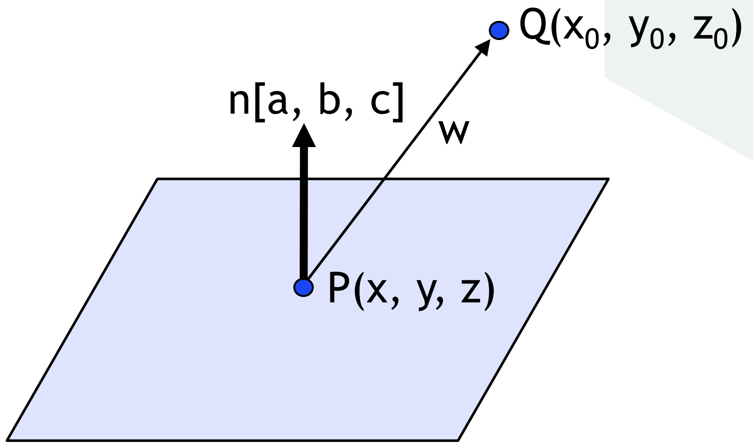

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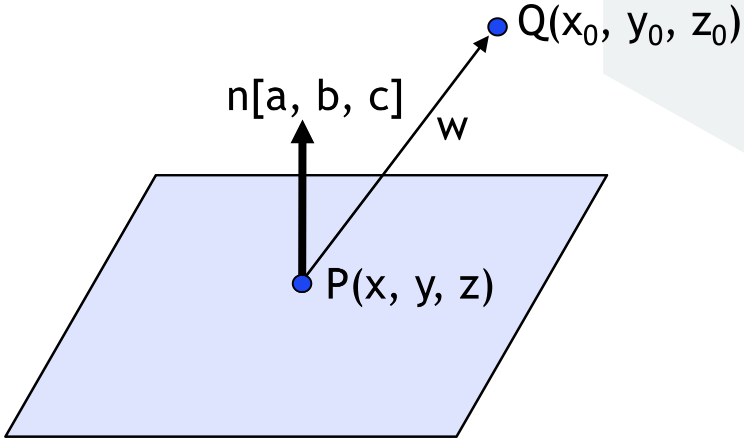

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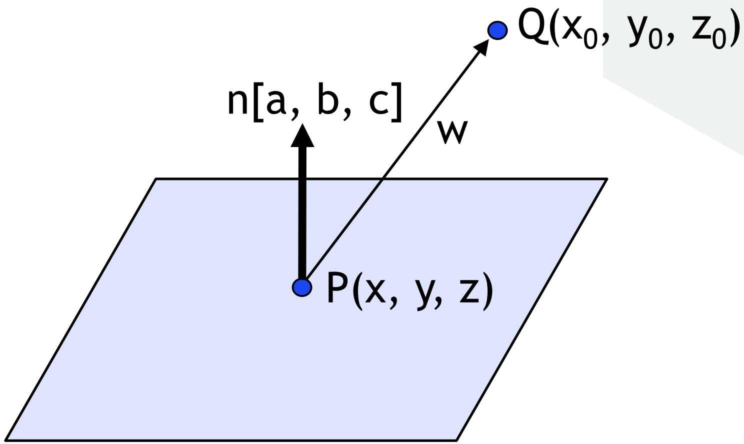
$$w = Q - P = [x_0 - x, y_0 - y, z_0 - z]$$

$$D = \frac{|n \cdot w|}{\|n\|}$$

$$= \frac{|a(x_0 - x) + b(y_0 - y) + c(z_0 - z)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}$$

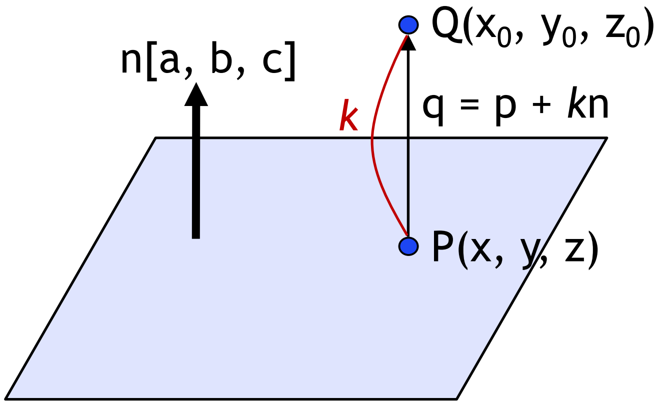


Closest Point on the Plane

- Find a point P on the plane (n, d) closest to one point Q in space.
  - $p = q - kn$ (*k is the shortest signed distance from point Q to the plane*)
  - *If n is a uint,*

    $k = n \cdot q + d$

    $p = q - (n \cdot q + d)n$



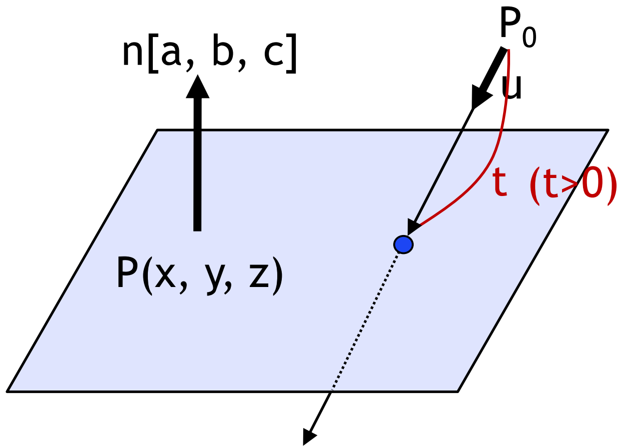
$$Distance(q, plane) = \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}$$

$where\ q(x_0, y_0, z_0)\ and\ Plane\ ax + by + cz + d = 0$

$Distance(q, plane) = n \cdot q + d\ (n\ is\ a\ unit\ vector)$

Intersection of Ray and Plane

- *Ray* $p(t) = p_0 + tu$ & *plane* $p \cdot n + d = 0$



Intersection of Ray and Plane

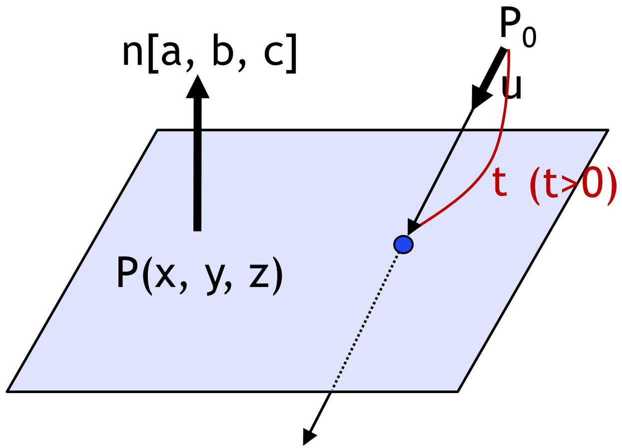
- *Ray* $p(t) = p_0 + tu$ & *plane* $p \cdot n + d = 0$
- Ray/Plane intersection :

$$(p_0 + tu) \cdot n + d = 0$$

$$p_0 \cdot n + tu \cdot n + d = 0$$

$$tu \cdot n = -d - p_0 \cdot n$$

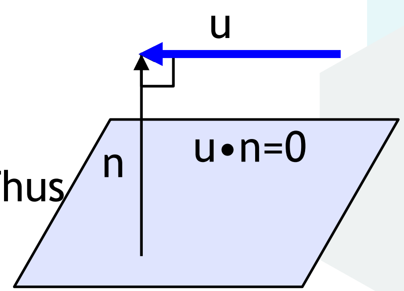
$$t = \frac{-(p_0 \cdot n + d)}{u \cdot n}$$



Intersection of Ray and Plane

$$t = \frac{-(p_0 \cdot n + d)}{u \cdot n}$$

- If the ray is parallel to the plane, the denominator u•n=0. Thus the ray does not intersect the plane.



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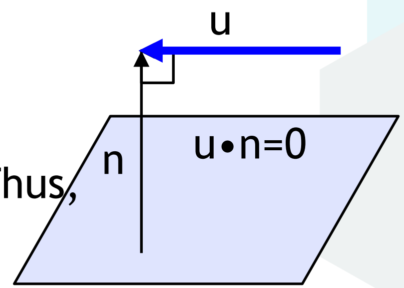
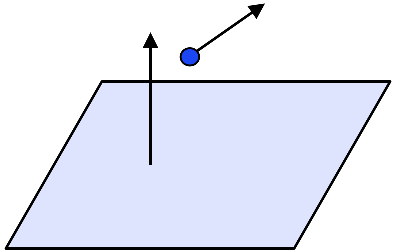


- If the value of t is not in the range [0, ∞), the ray does not intersect the plane.



$$p\left(\frac{-(p_0 \cdot n + d)}{u \cdot n}\right) = p_0 + \frac{-(p_0 \cdot n + d)}{u \cdot n} u$$

Matrix

- Matrix M (r×c matrix)
  - **Row** of horizontally arranged matrix elements
  - **Column** of vertically arranged matrix elements
  - M_{ij} is the **element** in row i and column j

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

r(2) rows

c(2) columns

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Matrix

2×5 matrix

$$\begin{pmatrix} 2 & -4 & 7 & 7/8 & 8 \\ -3 & 4 & 3/8 & 0 & 1 \end{pmatrix}$$

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$$\begin{pmatrix} 2 & -4 & 7 & 7/8 & 8 \\ -3 & 4 & 3/8 & 0 & 1 \end{pmatrix}$$

$m_{11} = 2$

$m_{12} = -4$

4×3 matrix

$$\begin{pmatrix} 4 & 0 & 12 \\ -5 & 4 & 3 \\ 12 & 3/8 & -1 \\ 1/2 & 18 & 0 \end{pmatrix}$$

Square Matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

Nondiagonal elements

Diagonal elements

- The n×n matrix is called an n-th square matrix. e.g. 2x2, 3x3, 4x4
- Diagonal elements vs. Non-diagonal elements

Identity Matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The identity matrix is expressed as I.
- All of the diagonals are 1, the remaining elements are 0 in **n x n** square matrix.
- M I = I M = M

Vectors as Matrices

- The n-dimension vector is expressed as a $1 \times \mathbf{n}$ matrix or an $\mathbf{n} \times 1$ matrix.
  - $1 \times n$ matrix is a row vector (also called a row matrix)
  - $n \times 1$ matrix is a column vector (also called a column matrix)

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix}$$

Vectors as Matrices

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Transpose Matrix

- Transpose of M (rxc matrix) is denoted by M^T and is converted to c×r matrix.
  - $M^T_{ij} = M_{ji}$
  - $(M^T)^T = M$
  - $D^T = D$ for any diagonal matrix D.

$$\begin{pmatrix} a & m & c \\ d & e & f \\ g & h & i \end{pmatrix}^T = \begin{pmatrix} a & d & g \\ m & e & h \\ c & f & i \end{pmatrix}$$

Transposing Matrix

$$\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^T = \begin{pmatrix} x & y & z \end{pmatrix}$$

Transpose Matrix

- Multiplying a matrix M with a scalar $\alpha = \alpha$M

$$\alpha M = \alpha \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \begin{pmatrix} \alpha m_{11} & \alpha m_{12} & \alpha m_{13} \\ \alpha m_{21} & \alpha m_{22} & \alpha m_{23} \\ \alpha m_{31} & \alpha m_{32} & \alpha m_{33} \end{pmatrix}$$

Two Matrices Addition

- Matrix C is the addition of A ($r \times c$ matrix) and B ($r \times c$ matrix), which is a $r \times c$ matrix.
- Each element c_{ij} is the sum of the ij^{th} element of A and the ij^{th} element of B.
- $c_{ij} = a_{ij} + b_{ij}$

Two Matrices Addition

- Matrix C is the addition of A (r×c matrix) and B (r×c matrix), which is a r×c matrix.

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$$\underset{r\times c}{\begin{pmatrix} 1 & 3 & 6 \\ 10 & 0 & -5 \\ 4 & 7 & 2 \end{pmatrix}} + \underset{r\times c}{\begin{pmatrix} 3 & 7 & 1 \\ 6 & 4 & 9 \\ 8 & -9 & 4 \end{pmatrix}} = \underset{r\times c}{\begin{pmatrix} 4 & 10 & 7 \\ 16 & 4 & 4 \\ 12 & -2 & 6 \end{pmatrix}}$$

Two Matrices Multiplication

- Matrix C(r×c matrix) is the product of A (r×n matrix) and B (n×c matrix).
- Each element c_{ij} is the vector dot product of the i^{th} row of A and the j^{th} column of B.
- $$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Two Matrices Multiplication

- Matrix C(r×c matrix) is the product of A (r×n matrix) and B (n×c matrix).
- Each element c_{ij} is the vector dot product of the i^{th} row of A and the j^{th} column of B.
- $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

3 + 18 + 48

$$\begin{pmatrix} 1 & 3 & 6 \\ 10 & 0 & -5 \\ 4 & 7 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 7 & 1 \\ 6 & 4 & 9 \\ 8 & -9 & 4 \end{pmatrix} = \begin{pmatrix} 69 & -35 & 52 \\ -10 & 115 & -10 \\ 70 & 38 & 75 \end{pmatrix}$$

r×n — n×c — r×c

must match

columns in result

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$7 + 12 + (-54)$

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Multiplying Two Matrices

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \end{pmatrix}$$

4×5 4×2 2×5

$$c_{24} = a_{21}m_{14}+a_{22}m_{24}$$

Matrix Operation

- $MI = IM = M$ (I is identity matrix)
- $A + B = B + A$: matrix addition commutative law
- $A + (B + C) = (A + B) + C$: matrix addition associative law
- $AB \neq BA$: Not hold matrix product commutative law
- $(AB)C = A(BC)$: matrix product associative law
- $ABCDEF = ((((AB)C)D)E)F = A((((BC)D)E)F) = (AB)(CD)(EF)$

Matrix Operation

- $\alpha(AB) = (\alpha A)B = A(\alpha B)$: Scalar-matrix product
- $\alpha(\beta A) = (\alpha\beta)A$
- $(vA)B = v(AB)$
- $(AB)^T = B^T A^T$
- $(M_1M_2M_3 \ldots M_{n-1}M_n)^T = M_n^TM_{n-1}^T \ldots M_3^TM_2^TM_1^T$

Matrix Determinant

- The determinant of a square matrix M is denoted by **|M|** or "**det M**".
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Matrix Determinant

- Inverse of M (square matrix) is denoted by M^{-1}.
- $M^{-1} = \frac{adjM}{|M|}$
- $(M^{-1})^{-1} = M$
- $M(M^{-1}) = M^{-1}M = 1$
- The determinant of a non-singular matrix (i.e, invertible) is nonzero.
- The adjoint of M, denoted "**adj M**" is **the transpose of the matrix of cofactors**.

$$adjM = \begin{pmatrix} 1 & 3 & 6 \\ 10 & 0 & -5 \\ 4 & 7 & 2 \end{pmatrix}^T$$

Cofactor of a Square Matrix & Computing Determinant using Cofactor

- Cofactor of a square matrix M at a given row and column is the signed determinant of the corresponding Minor of M.
- $C_{ij} = (-1)^{ij} \left| M^{\{ij\}} \right|$

Cofactor of a Square Matrix
& Computing Determinant using Cofactor

Calculation of n×n determinant using cofactor :

$$|M| = \sum_{j=1}^{n} m_{ij} c_{ij} = \sum_{j=1}^{n} m_{ij} (-1)^{i+j} \left| M^{\{ij\}} \right|$$

$$|M| = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} = m_{11} \begin{pmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{pmatrix}$$

$$-m_{12} \left| M^{\{12\}} \right|$$

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$$+m_{13} \left| M^{\{13\}} \right|$$

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Minor of a Matrix

- The submatrix $M^{\{ij\}}$ is known as a minor of M, obtained by deleting row i and column j from M.

$$M = \begin{pmatrix} -4 & -3 & 3 \\ 0 & 2 & -2 \\ 1 & 4 & -1 \end{pmatrix} \qquad M^{\{12\}} = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$$

Determinant, Cofactor, Inverse Matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$\det M = m_{11}m_{22} - m_{12}m_{21}$$

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$$\operatorname{adj} M = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \qquad M^{-1} = \frac{1}{detM}\begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

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$$adjM = \begin{pmatrix} (m_{22}m_{33} - m_{23}m_{32}) & -(m_{21}m_{33} - m_{13}m_{32}) & (m_{21}m_{23} - m_{22}m_{13}) \\ -(m_{21}m_{33} - m_{23}m_{31}) & (m_{11}m_{33} - m_{13}m_{31}) & -(m_{11}m_{23} - m_{13}m_{21}) \\ (m_{21}m_{32} - m_{22}m_{31}) & -(m_{11}m_{32} - m_{21}m_{31}) & (m_{11}m_{22} - m_{12}m_{21}) \end{pmatrix}$$

$$M^{-1} = \frac{adjM}{detM}$$

Multiplying a Vector and a Matrix

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{pmatrix}$$

$$= \begin{pmatrix} xp_x + yq_x + zr_x & xp_y + yq_y + zr_y & xp_z + yq_z + zr_z \end{pmatrix}$$

$$= xp + yq + zr$$

- A coordinate space transformation can be expressed using a vector-matrix product.
  - **uM** = **v** // matrix M converts vector u to vector v

Multiplying a Vector and a Matrix

- Vector-matrix multiplication in OpenGL (Column-Major Order)
  - **v** = **M** * **u** // matrix M converts vector u to vector v

v = M * u

$$\begin{pmatrix} xm_{11}+ym_{12}+zm_{13} \\ xm_{21}+ym_{22}+zm_{23} \\ xm_{31}+ym_{32}+zm_{33} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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