Geometric Objects and Transformation

321190
2007년 봄학기
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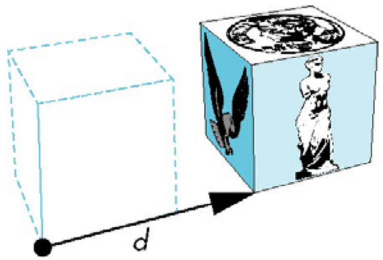
박경신

3D Transformations

- 일반적으로 3차원 변환은 2차원 변환의 확장으로 생각하면 된다.
- 변환 행렬은 각각 한 행과 열이 추가된다.
- 3차원의 이동 (Translation), 크기변환 (Scaling), 밀림변환 (Shearing)의 기본 원리는 2차원과 같다.
- 그러나, 3차원의 회전은 좀 더 복잡하다.

3D Translation

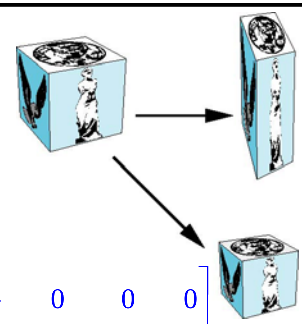
$$p' = p + d \quad p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \quad d = \begin{bmatrix} dx \\ dy \\ dz \\ 0 \end{bmatrix}$$



$$p' = Tp \quad T = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -dx \\ 0 & 1 & 0 & -dy \\ 0 & 0 & 1 & -dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

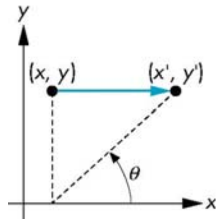
3D Scale

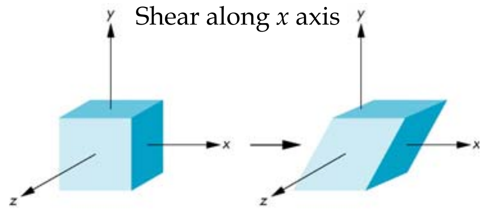
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$p' = Sp \quad S = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} \frac{1}{sx} & 0 & 0 & 0 \\ 0 & \frac{1}{sy} & 0 & 0 \\ 0 & 0 & \frac{1}{sz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Shear





x′ = x + y cot θ
y′ = y
z′ = z

$$\mathbf{H}_{xy}(\theta) = \begin{bmatrix} 1 & \cot\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

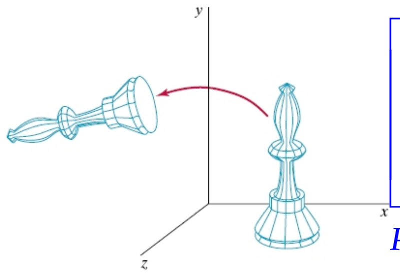
3D Rotation

- Z-축으로 회전

  x′ = x cosθ – y sinθ
  y′ = x sinθ + y cosθ
  z′ = z

$$R^{-1}(\theta) = R(-\theta)$$

$$R^{-1}(\theta) = R^{T}(\theta)$$



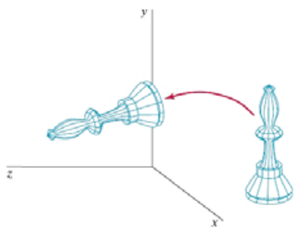
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_Z(\theta) \cdot P$$

3D Rotation

- X-축으로 회전

  y′ = y cosθ – z sinθ
  z′ = y sinθ + z cosθ
  x′ = x



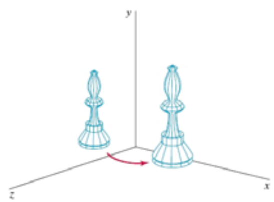
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_X(\theta) \cdot P$$

3D Rotation

- Y-축으로 회전

  x′ = x cosθ + z sinθ
  z′ = -x sinθ + z cosθ
  y′ = y



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

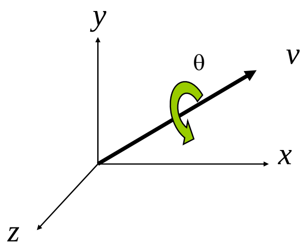
$$P' = R_Y(\theta) \cdot P$$

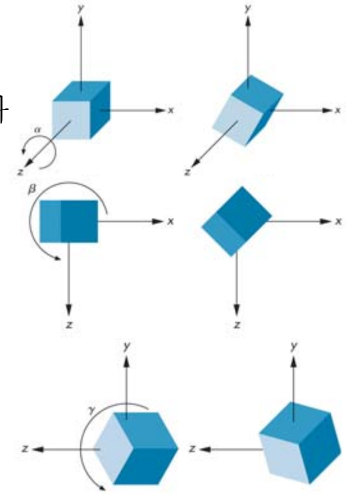
3D Rotation about the Origin

- 원점(0, 0, 0)에서 임의의 회전은 세 축에 대한 세 번의 연속적인 회전으로 구성할 수 있다.

  $R(\theta) = R_Z(\theta_Z)R_Y(\theta_Y)R_X(\theta_X)$

  θ_X, θ_Y, θ_Z를 오일러 앵글이라 부른다

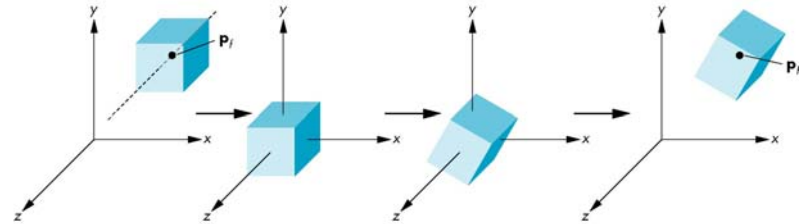




Rotation About a Pivot other than the Origin

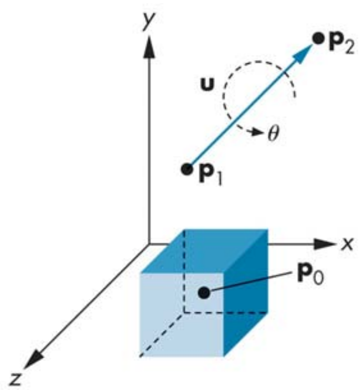
- Pivot (P_f)을 원점(0, 0, 0)으로 이동 후, 회전 후, 다시 Pivot으로 이동한다.
- **M** = **T**(p_f) **R**$_Z$(θ) **T**($-p_f$)

$$M = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & x_f - x_f\cos\theta + y_f\sin\theta \\ \sin\theta & \cos\theta & 0 & y_f - x_f\sin\theta - y_f\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3D Rotation about an Arbitrary Axis

- P_0 를 원점으로 이동한다.
- 임의의 축 u를 z-축에 정렬 (align) 시키기 위해 두 번 회전을 수행한다.
- Z-축으로 θ만큼 회전한다.
- 두 번의 회전을 되돌린다 (undo alignment).
- 다시 P_0 로 이동한다.



$$M = T(P_0)R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x)T(-P_0)$$

3D Rotation about an Arbitrary Axis

- The translation matrix, $T(-P_0)$

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

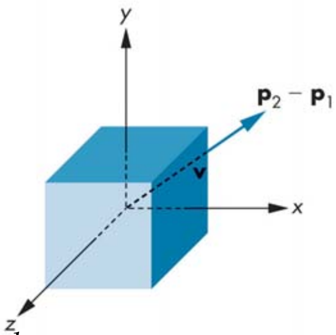
3D Rotation about an Arbitrary Axis

- The rotation-axis vector

  $u = P_2 - P_1$

  $= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
- Normalize u:

$$v = \frac{u}{\|u\|} = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$



- Rotate along x-axis until *v* hits *xz-plane*
- Rotate along y-axis until *v* hits *z-axis*

3D Rotation about an Arbitrary Axis

- Find θ_x and θ_y

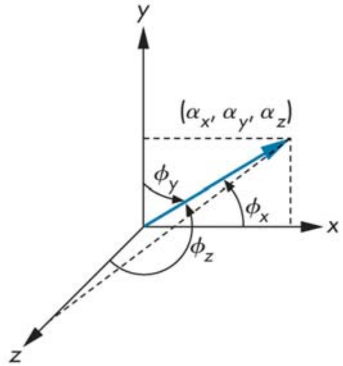
  $v = (\alpha_x, \alpha_y, \alpha_z)$

  $\alpha_x^2 + \alpha_y^2 + \alpha_z^2 = 1$
- Direction cosines:

$$\cos\phi_x = \alpha_x$$

$$\cos\phi_y = \alpha_y$$

$$\cos\phi_z = \alpha_z$$

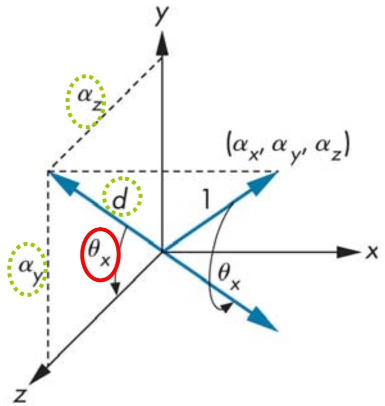


3D Rotation about an Arbitrary Axis

- Compute x-rotation θ_x

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha_z}{d} & -\frac{\alpha_y}{d} & 0 \\ 0 & \frac{\alpha_y}{d} & \frac{\alpha_z}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$

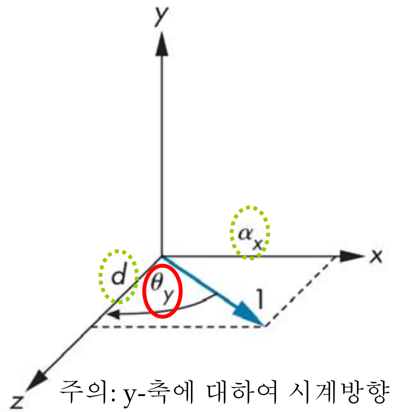


3D Rotation about an Arbitrary Axis

- Compute y-rotation θ_y

$$R_y(\theta_y) = \begin{bmatrix} d & 0 & -\alpha_x & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$



주의: y-축에 대하여 시계방향

3D Rotation about an Arbitrary Axis

- Rotation about the z axis

$$R_z(\theta_z) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Undo alignment, $R_x(-\theta_x)R_y(-\theta_y)$
- Undo translation, $T(P_0)$

- $M = T(P_0)R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x)T(-P_0)$

3D Rotation about an Arbitrary Axis

- 회전축(axis)/각(angle)로부터 다음과 같은 회전행렬 (rotation matrix)을 만든다.

$$R = \mathbf{I}\cos\theta + \mathbf{Symmetric}\ (1-\cos\theta) + \mathbf{Skew}\sin\theta$$

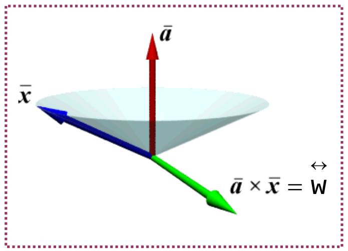
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cos\theta + \begin{bmatrix} a_x^2 & a_xa_y & a_xa_z \\ a_xa_y & a_y^2 & a_ya_z \\ a_xa_z & a_ya_z & a_z^2 \end{bmatrix}(1-\cos\theta) + \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}\sin\theta$$

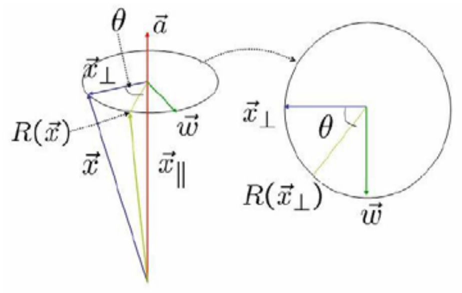
$$= \begin{bmatrix} a_x^2+\cos\theta(1-a_x^2) & a_xa_y(1-\cos\theta)-a_z\sin\theta & a_xa_z(1-\cos\theta)+a_y\sin\theta \\ a_xa_y(1-\cos\theta)+a_z\sin\theta & a_y^2+\cos\theta(1-a_y^2) & a_ya_z(1-\cos\theta)-a_x\sin\theta \\ a_xa_z(1-\cos\theta)-a_y\sin\theta & a_ya_z(1-\cos\theta)+a_x\sin\theta & a_z^2+\cos\theta(1-a_z^2) \end{bmatrix}$$

3D Rotation as Vector Components

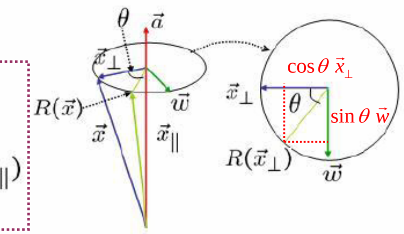
- Rotate a point by θ about an arbitrary axis, a =[a_x, a_y, a_z]

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \left(\mathbf{Symmetric}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right)(1-\cos\theta) + \mathbf{Skew}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right)\sin\theta + \mathbf{I}\cos\theta \right)\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$





3D Rotation as Vector Components



$$\begin{aligned} \vec{w} &= \vec{a} \times \vec{x}_\perp \\ &= \vec{a} \times (\vec{x} - \vec{x}_\parallel) \\ &= (\vec{a} \times \vec{x}) - (\vec{a} \times \vec{x}_\parallel) \\ &= \vec{a} \times \vec{x} \end{aligned}$$

$$R(\vec{x}_\perp) = \cos\theta\,\vec{x}_\perp + \sin\theta\,\vec{w}$$

$$\begin{aligned} \vec{x}_\parallel &= (\vec{a}\cdot\vec{x})\vec{a} \\ \vec{x}_\perp &= \vec{x} - \vec{x}_\parallel = \vec{x} - (\vec{a}\cdot\vec{x})\vec{a} \end{aligned}$$

$$\begin{aligned} R(\vec{x}) &= R(\vec{x}_\parallel) + R(\vec{x}_\perp) \\ &= R(\vec{x}_\parallel) + \cos\theta\,\vec{x}_\perp + \sin\theta\,\vec{w} \\ &= (\vec{a}\cdot\vec{x})\vec{a} + \cos\theta(\vec{x} - (\vec{a}\cdot\vec{x})\vec{a}) + \sin\theta\,\vec{w} \\ &= \cos\theta\,\vec{x} + (1-\cos\theta)(\vec{a}\cdot\vec{x})\vec{a} + \sin\theta(\vec{a}\times\vec{x}) \end{aligned}$$

3D Rotation as Vector Components

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \left(\textbf{Symmetric}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right)(1-\cos\theta) + \textbf{Skew}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right)\sin\theta + \mathbf{I}\cos\theta \right)\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- The vector a specifies the axis of rotation. This axis vector must be normalized.
- The rotation angle is given by θ.
- The basic idea is that *any rotation can be decomposed into weighted contributions from three different vectors.*

3D Rotation as Vector Components

- *The symmetric matrix of a vector* generates a vector in the direction of the axis.
- The symmetric matrix is composed of the outer product of a row vector and an column vector of the same value.

$$\text{Symmetric}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right) = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\begin{bmatrix} a_x & a_y & a_z \end{bmatrix} = \begin{bmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{bmatrix}$$

$$\text{Symmetric}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \bar{a}(\bar{a}\cdot\bar{x})$$

3D Rotation as Vector Components

- *Skew symmetric matrix of a vector* generates a vector that is perpendicular to both the axis and it's input vector.

$$\text{Skew}\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}\right) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\text{Skew}(\bar{a})\bar{x} = \bar{a}\times\bar{x}$$

3D Rotation as Vector Components

- First, consider a rotation by 0. :

$$Rotate\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, 0\right) = \begin{bmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{bmatrix}(1-1) + \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}0 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- For instance, a rotation about the x-axis:

$$Rotate\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \theta\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(1-\cos\theta) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\sin\theta + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cos\theta$$

$$Rotate\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \theta\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

3D Rotation as Vector Components

- For instance, a rotation about the y-axis:

$$Rotate\left(\begin{bmatrix}0\\1\\0\end{bmatrix},\theta\right)=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}(1-\cos\theta)+\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}\sin\theta+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cos\theta$$

$$Rotate\left(\begin{bmatrix}0\\1\\0\end{bmatrix},\theta\right)=\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix}$$

- For instance, a rotation about the z-axis:

$$Rotate\left(\begin{bmatrix}0\\0\\1\end{bmatrix},\theta\right)=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}(1-\cos\theta)+\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\sin\theta+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cos\theta$$

$$Rotate\left(\begin{bmatrix}0\\0\\1\end{bmatrix},\theta\right)=\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$$