Geometric Objects and Transformation

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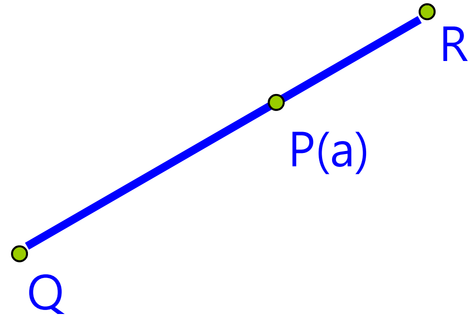
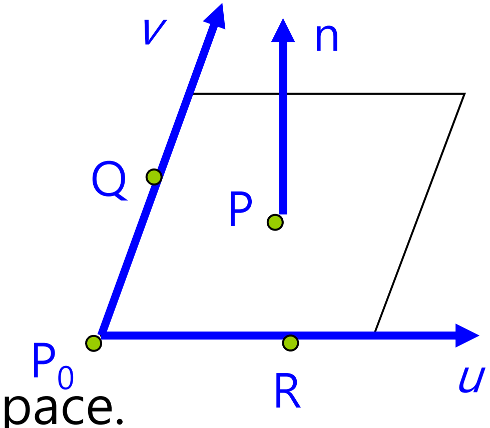
Geometric Objects

- Line
  - 2 points: R, Q
  - v = R- Q

    $P = Q + \alpha v = Q + \alpha(R - Q) = \alpha R + (1 - \alpha)Q$
  - $P = \alpha_1 R + \alpha_2 Q$ where $\alpha_1 + \alpha_2 = 1$ (affine sum)



- Plane
  - 3 points: P_0, Q, R
  - $T(\alpha, \beta) = P + \alpha u + \beta v$
  - $n \bullet (P - P_0) = 0$ where $n = u \times v$



- 3D objects
  - It is a set of vertices in three dimensional space.
  - It is described by the surface, and is hollow.
  - It can be composed of convex polygons.
  - An arbitrary polygon is divided into triangular polygons, i.e., tessellate.

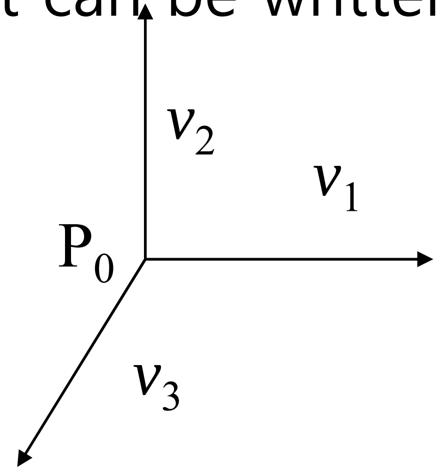
Coordinate Systems

- Consider a basis, $v_1, v_2, \ldots, v_n$
- Any vector v can be written as $v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$
- The list of scalars $\{a_1, a_2, \ldots a_n\}$ is the representation of v with respect to the given basis.
- We can write the representation as a row or column array of scalars.

$$\mathbf{a} = [\alpha_1 \;\; \alpha_2 \;\; \ldots \;\; \alpha_n]^T = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_n \end{bmatrix}$$

Frames

- The affine space contains points.
- If we work in an affine space we can add the origin to the basis vectors to form a **frame**.
- Frame: (P_0, v_1, v_2, v_3)
- Within this frame, every vector can be written as:
  $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$
- Every point can be written as: $P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$



$\mathbf{v} = [\alpha_1\ \alpha_2\ \alpha_3\ 0\ ]^T$

$\mathbf{p} = [\beta_1\ \beta_2\ \beta_3\ 1\ ]^T$

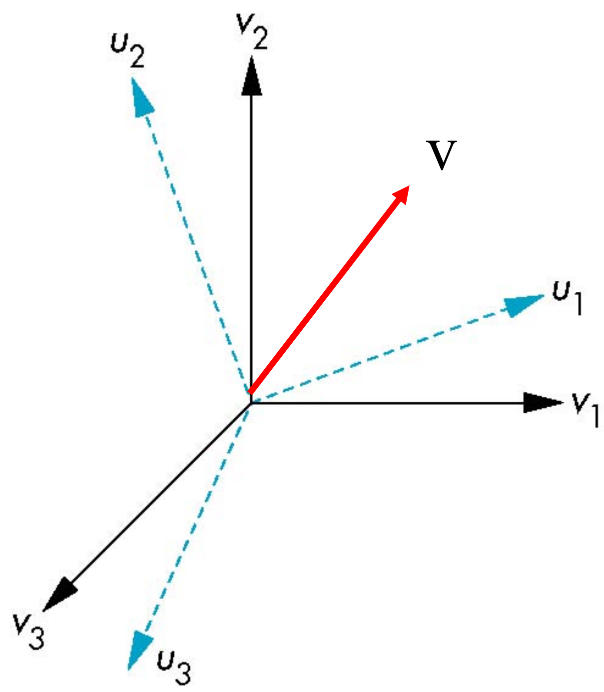
Change of Coordinate Systems

- Consider two representations of a the same vector, v, with respect to two different bases : $\{v_1, v_2, v_3\}$, $\{u_1, u_2, u_3\}$

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$
$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$
$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$

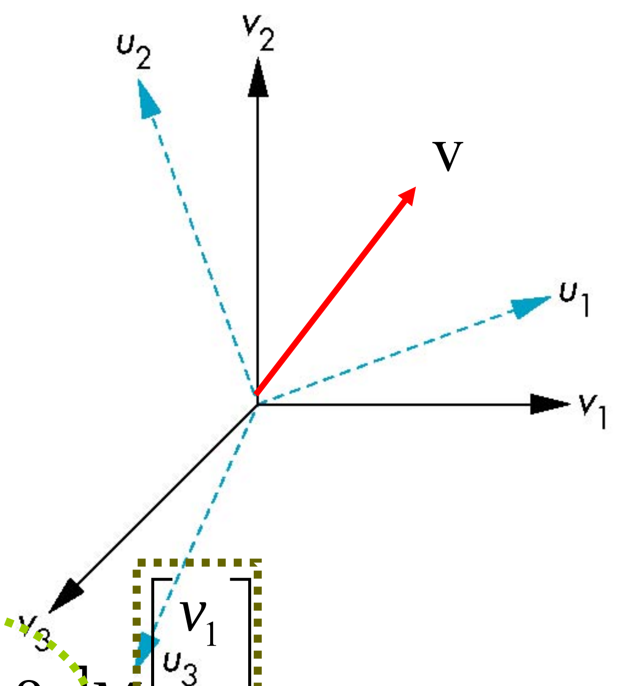
$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$



Change of Coordinate Systems

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \underbrace{[\alpha_1 \quad \alpha_2 \quad \alpha_3]}_{a^T} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$



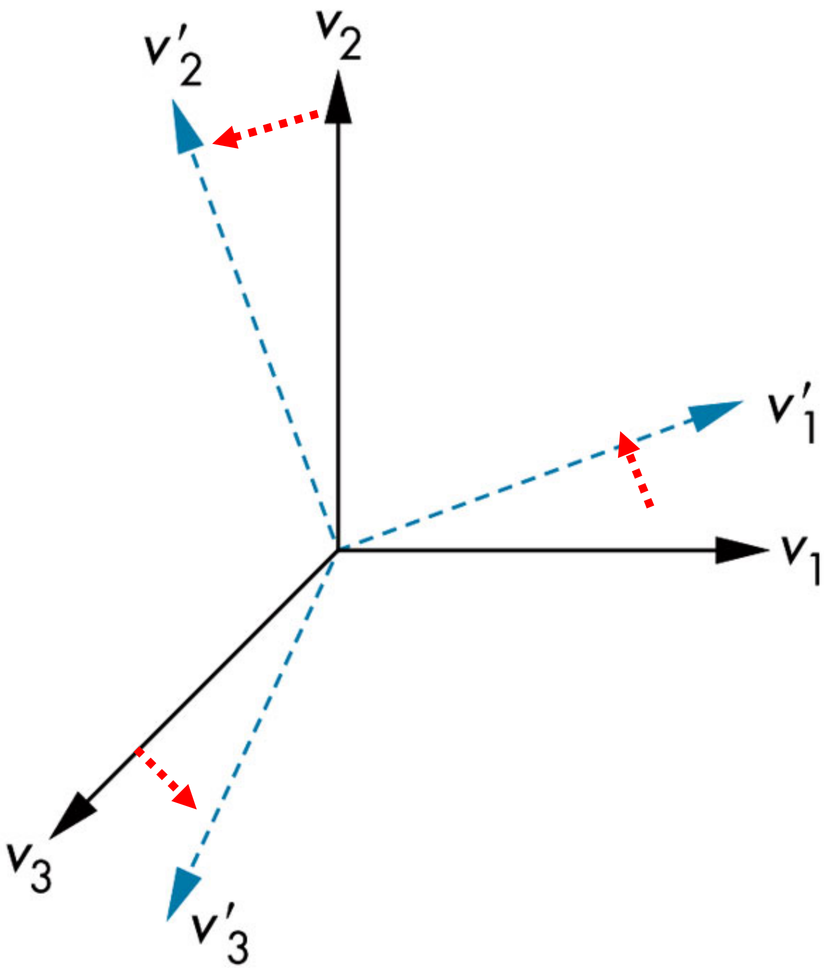
$$v = \underbrace{[\beta_1 \quad \beta_2 \quad \beta_3]}_{b^T} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$[\alpha_1 \quad \alpha_2 \quad \alpha_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = [\beta_1 \quad \beta_2 \quad \beta_3] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = [\beta_1 \quad \beta_2 \quad \beta_3] M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{a} = M^T \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{b} \quad \therefore \mathrm{a} = \mathrm{M}^T \mathrm{b} \quad \therefore \mathrm{b} = (\mathrm{M}^T)^{-1} \mathrm{a}$$

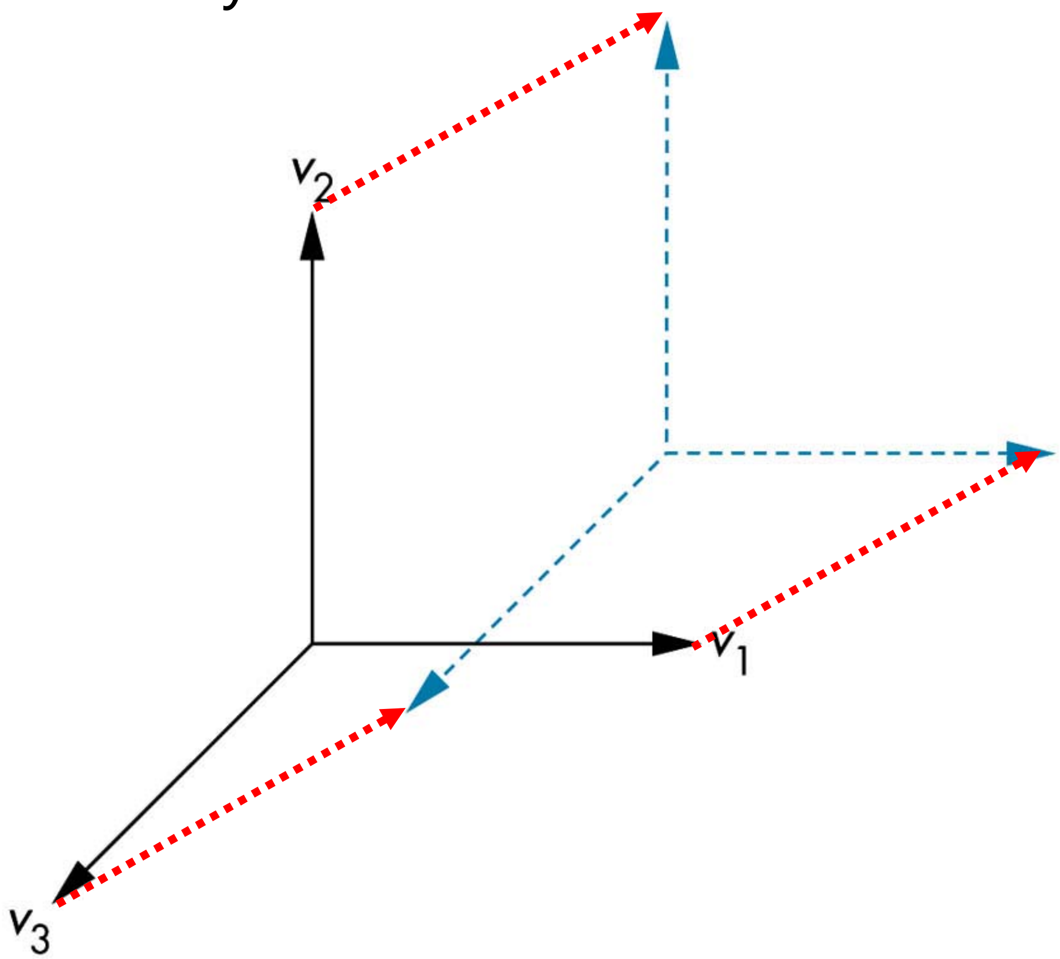
Rotation and Scaling of a Basis

- The rotation and scaling transformation can be represented by the basis vectors.



Translation of a Basis

- However, a simple translation of the origin is not represented by a 3x3 matrix.

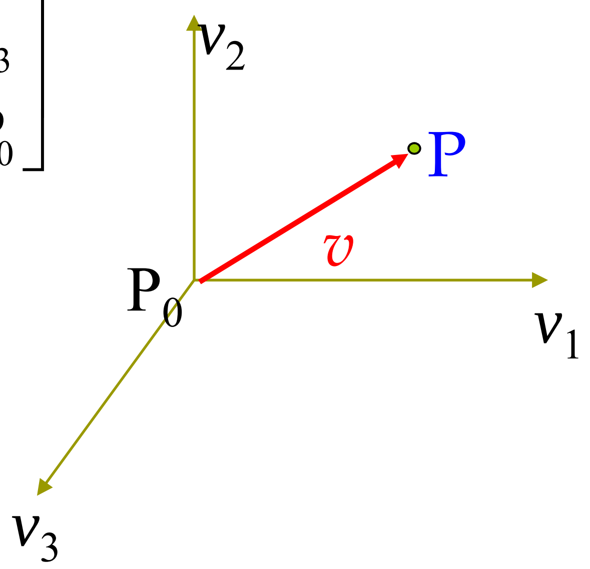


Homogeneous Coordinates

$$\textbf{vector}\ v = \sum \alpha_i v_i = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$\textbf{point}\ P = P_0 + \sum \alpha_i v_i = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$P = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix}, v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 0 \end{bmatrix}$$



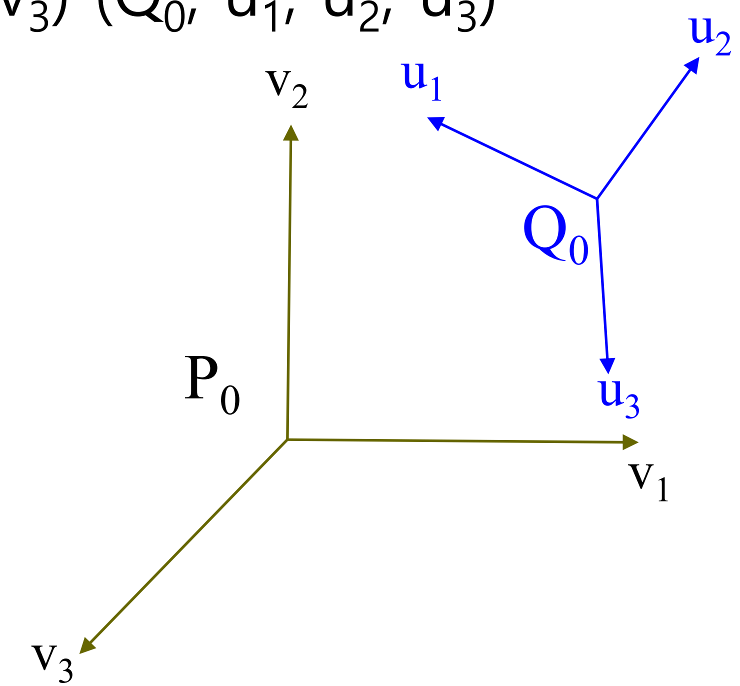
Change of Frames

- Consider two frames (P_0, v_1, v_2, v_3) (Q_0, u_1, u_2, u_3)

$$u_1 = \gamma_{11}v_1+\gamma_{12}v_2+\gamma_{13}v_3$$
$$u_2 = \gamma_{21}v_1+\gamma_{22}v_2+\gamma_{23}v_3$$
$$u_3 = \gamma_{31}v_1+\gamma_{32}v_2+\gamma_{33}v_3$$
$$Q_0 = \gamma_{41}v_1+\gamma_{42}v_2+\gamma_{43}v_3 +P_0$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$



Change of Frames

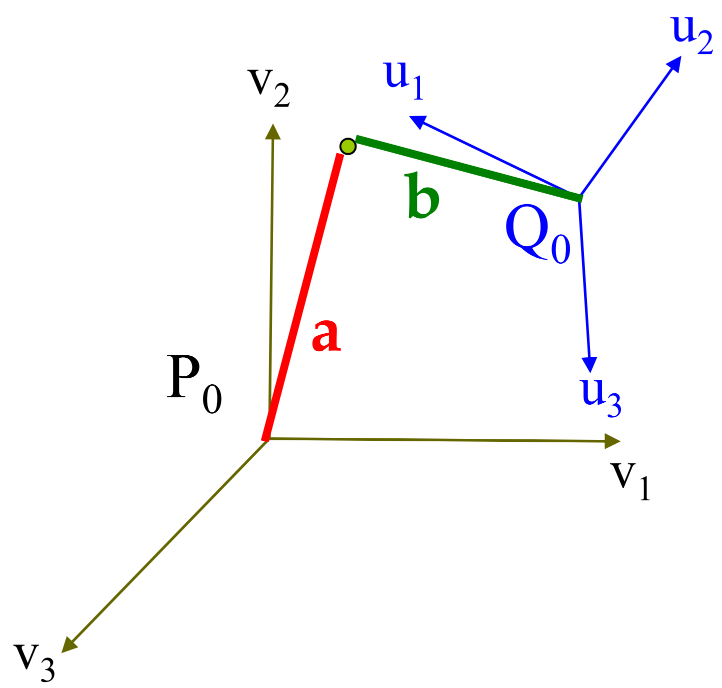
- Within the two frames (P_0, v_1, v_2, v_3) (Q_0, u_1, u_2, u_3) any point and vector has a representation of the same form

$$b^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = b^T \mathrm{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = a^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$\mathrm{M}^T = \begin{bmatrix} \gamma_{11} & \gamma_{21} & \gamma_{31} & \gamma_{41} \\ \gamma_{12} & \gamma_{22} & \gamma_{32} & \gamma_{42} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & \gamma_{43} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore a = \mathrm{M}^T b$$

$$\therefore b = (\mathrm{M}^T)^{-1} a$$



OpenGL Frames

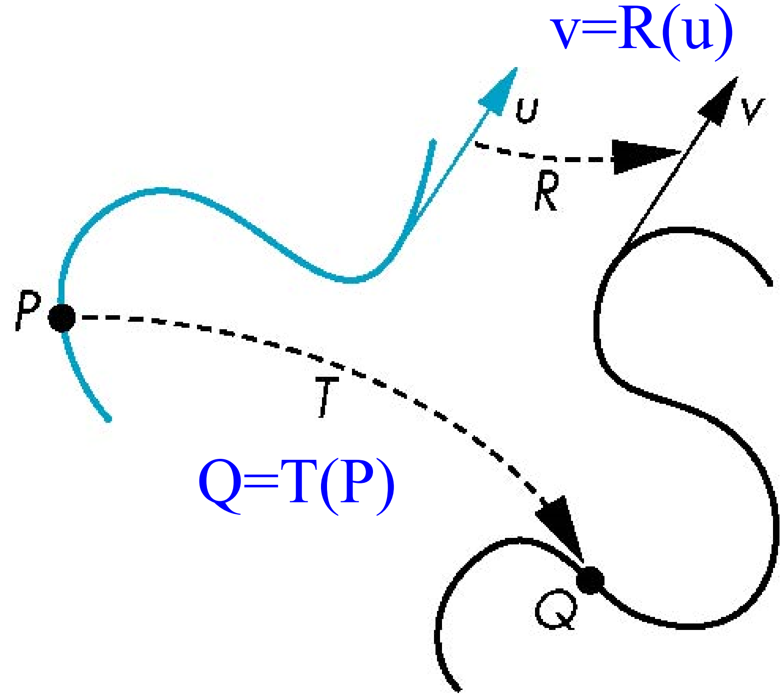
- Model-view coordinate system
- World coordinate system
- Camera coordinate system
- Clipping coordinate system
- Normalized device coordinate system
- Screen coordinate system

General Transformations

- A transformation maps points to other points and/or vectors to other vectors

$q = f(p)$

$v = f(u)$



Affine Transformations

- The affine transformation maintains collinearity.
  - That is, every affine transformation preserves lines. All points on a line exist on the transformed line.
- Also, it maintains the ratio of distance.
  - That is, the midpoint of a line is located at the midpoint of the transformed line segment.
- $P' = f(P)$
- $P' = f(\alpha P_1 + \beta P_2) = \alpha\, f(P_1) + \beta\, f(P_2)$

Affine Transformation

- Most transformation in computer graphics are affine transformation. Affine transformation include **translation, rotation, scaling, shearing**.
- The transformed point P′ (x′, y′, z′) can be expressed as a linear combination of the original point P (x, y, z), i.e.

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Affine Transformation

- The transformed point P′ (x′, y′, z′) can be expressed as a linear combination of the original point P (x, y, z), i.e.,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} \alpha_{11} x + \alpha_{12} y + \alpha_{13} \\ \alpha_{21} x + \alpha_{22} y + \alpha_{23} \\ 1 \end{pmatrix}$$

↓

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Geometric Transformation

- Geometric transformation refers to a function that transforms a group of points describing a geometric object to new points.
- At this time, the points are transformed to a new position while maintaining the relationship between the vertices of the objects.
- Basic transformation
  - Translation
  - Rotation
  - Scaling

OpenGL Column-Major Order

- **2D transformation matrix, M**

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- **If Point p is a column vector (OpenGL) :**

$$\mathbf{p}' = \mathbf{M}\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- If Point p is a row vector:

$$\mathbf{p}' = \mathbf{p}\mathbf{M}^{\mathrm{T}}$$

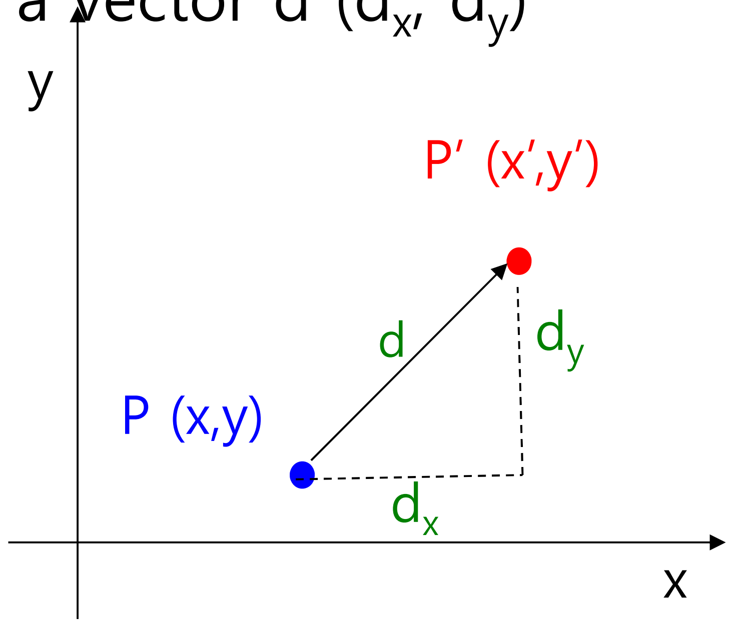
$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

2D Translation

- Translation moves a point P(x, y) to a new location P′(x′, y′)
- Displacement determined by a vector d (d_x, d_y)

  $x' = x + d_x$

  $y' = y + d_y$



$$P' = P + d \;\; where \;\; P' = \begin{pmatrix} x' \\ y' \end{pmatrix} P = \begin{pmatrix} x \\ y \end{pmatrix} d = \begin{pmatrix} d_x \\ d_y \end{pmatrix}$$

2D Translation

- What if you move an object with multiple vertices?

Translate individual vertices

2D Translation

- Using the homogeneous coordinate representation in some frame

  $\mathbf{p}=[\ x\ y\ 1]^T$

  $\mathbf{p}'=[x'\ y'\ 1]^T$

  $\mathbf{d}=[dx\ dy\ 0]^T$

- Hence $\mathbf{p}' = \mathbf{p} + \mathbf{d}$ or

  $x'=x+d_x$

  $y'=y+d_y$

Note that this expression is in four dimensions and expresses point = vector + point

2D Translation

- We can also express 2D translation using a 3 x 3 matrix $\mathbf{T}$ in homogeneous coordinates:

$\mathbf{p}' = \mathbf{T}\mathbf{p}$ where

$$\mathbf{T} = \mathbf{T}(d_x, d_y) = \begin{pmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{pmatrix}$$

- This form is better for implementation because all affine transformations can be expressed this way and multiple transformations can be concatenated together.

2D Translation

- **2D translation**

  $$x' = x + d_x$$
  $$y' = y + d_y$$

- **Inverse translation**

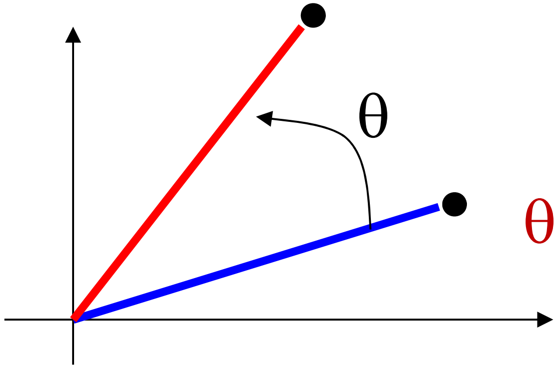
  $$x = x' - d_x$$
  $$y = y' - d_y$$

- **Identity translation**

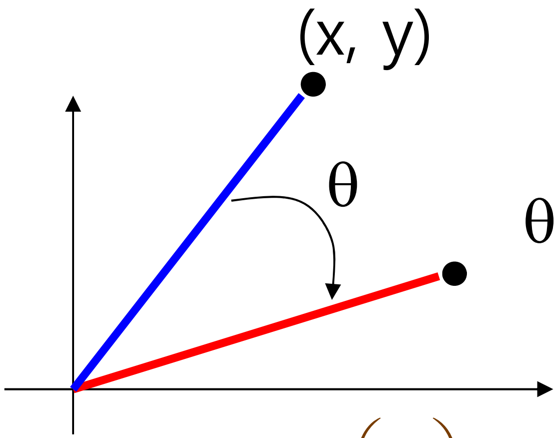
  $$x' = x + 0$$
  $$y' = y + 0$$

Rotation

- 2D rotation about the origin by θ



θ > 0 : Rotate counter clockwise in RHS



θ < 0 : Rotate clockwise in LHS

$$\theta = \arctan\left(\frac{y}{x}\right)$$

2D Rotation

- Rotation of a point P(x,y) by θ about an origin (0,0)

$x = r\cos(\phi) \qquad y = r\sin(\phi)$

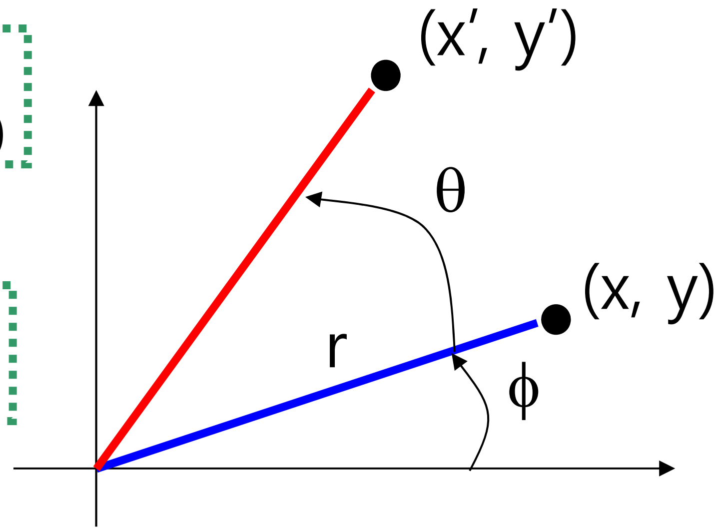
$x' = r\cos(\phi + \theta) \qquad y' = r\sin(\phi + \theta)$

$$\begin{aligned} x' &= r\cos(\phi + \theta) \\ &= r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta) \\ &= x\cos(\theta) - y\sin(\theta) \end{aligned}$$

$$\begin{aligned} y' &= r\sin(\phi + \ ) \\ &= r\sin(\phi)\cos(\theta) + r\cos(\phi)\sin(\theta) \\ &= y\cos(\theta) + x\sin(\theta) \end{aligned}$$

$x' = x\cos(\theta) - y\sin(\theta)$

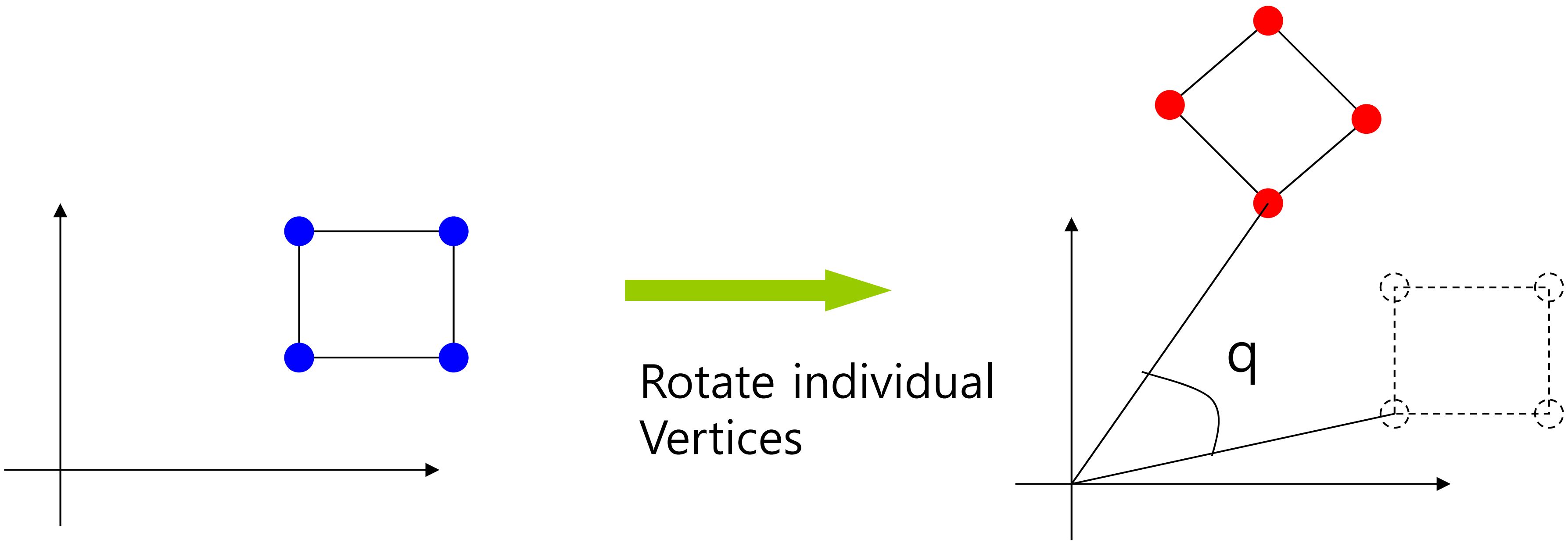
$y' = y\cos(\theta) + x\sin(\theta)$



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2D Rotation

- What if you rotate an object with multiple vertices?



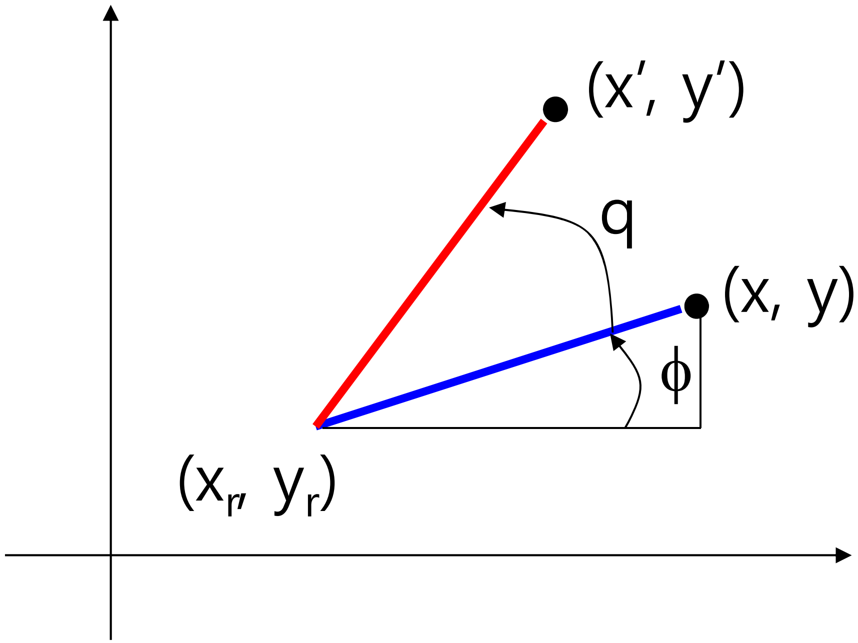
2D Rotation about an Arbitrary Pivot

- Rotation of a point P(x,y) by θ about an arbitrary pivot point, (x_r, y_r) :

  $P' = R(\theta)\ P$

  $x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$

  $y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$



2D Rotation

- **2D rotation**

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- **Inverse rotation**

$$R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

- **Identity rotation**

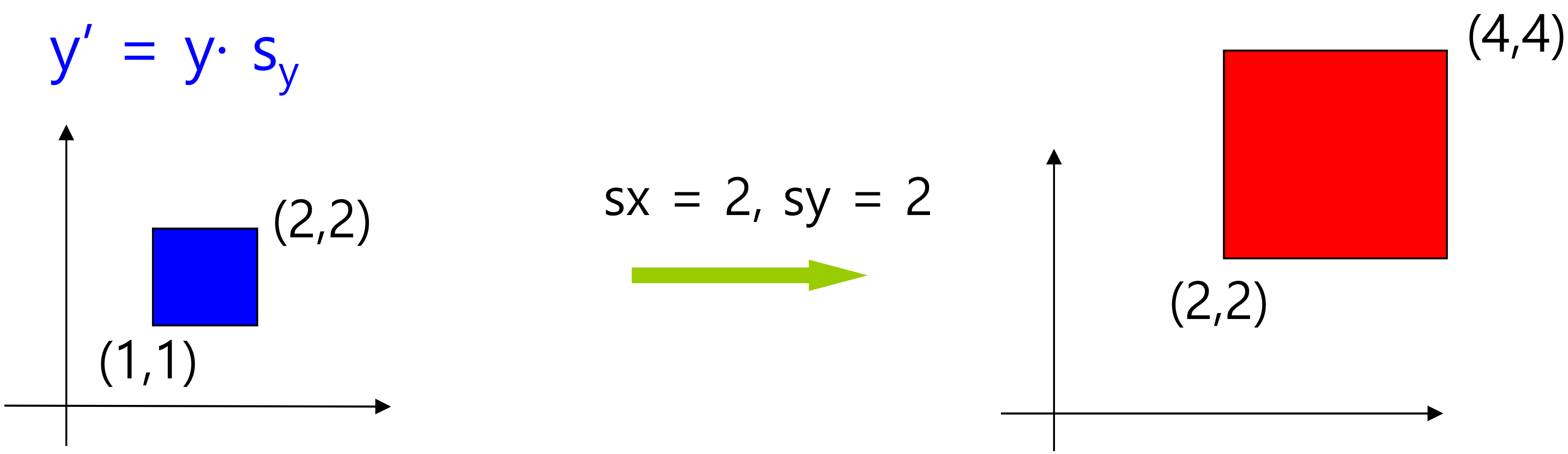
$$R_{\theta=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2D Scale

- Scaling makes an object larger or smaller by a scaling factor (s_x, s_y). This is affine non-rigid-body transformation. Scaling by 1 does not change an object.
- Scaling is done by an origin. Scaling changes not only the size of object, but also the position of object.

$$x' = x \cdot s_x$$

$$y' = y \cdot s_y$$



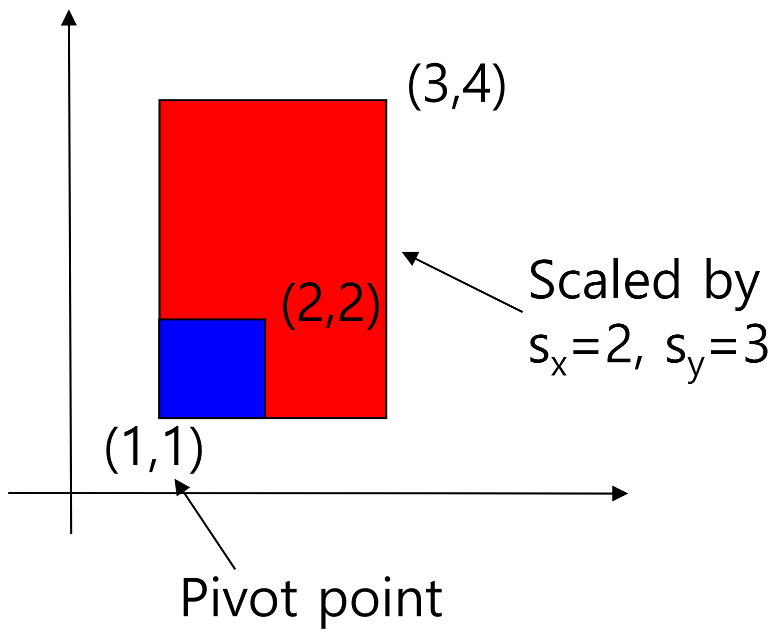
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2D Scale about an Arbitrary Pivot

- Scale a point P(x,y) by a scaling factor relative to an arbitrary pivot point, (x_f, y_f) : $P' = S(s_x, s_y)\ P$

$$x' = x_f + (x - x_f)\ s_x$$
$$y' = y_f + (y - y_f)\ s_y$$

$$x' = x\ s_x + x_f\ (1 - s_x)$$
$$y' = y\ s_y + y_f\ (1 - s_y)$$



2D Scale

- **2D scale**

$$S = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$$

- **Inverse scale**

$$S^{-1} = \begin{pmatrix} 1/s_x & 0 \\ 0 & 1/s_y \end{pmatrix}$$

- **Identity scale**

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2D Reflection (Mirror)

- Reflection is the transformation of an object in opposite direction with respect to a fixed point.
  - Reflection preserves angles and lengths.
- 2D reflection over x axis

  $x' = x$

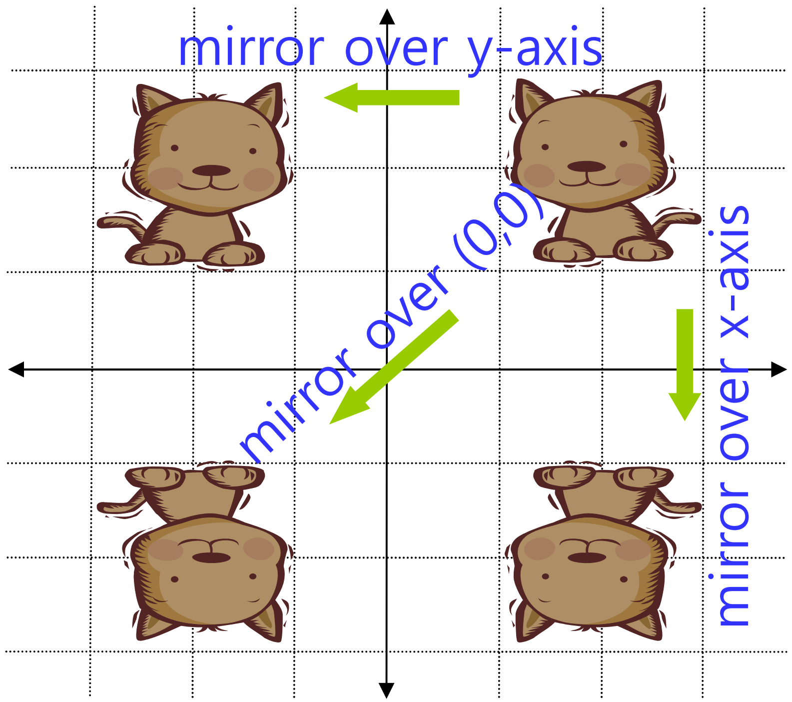
  $y' = -y$
- 2D reflection over y axis

  $x' = -x$

  $y' = y$
- 2D reflection over (0,0)

  $x' = -x$

  $y' = -y$



2D Reflection (Mirror)

- 2D reflection over a line, y = x

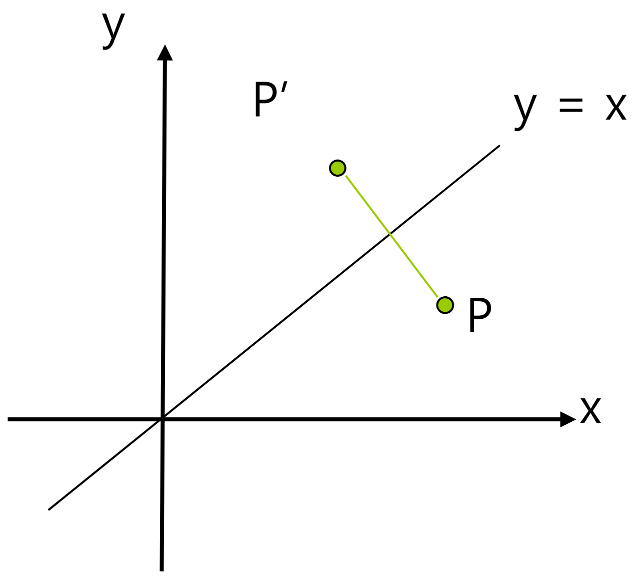
  $x' = y$

  $y' = x$

- 2D reflection over a line, y = -x

  $x' = -y$

  $y' = -x$



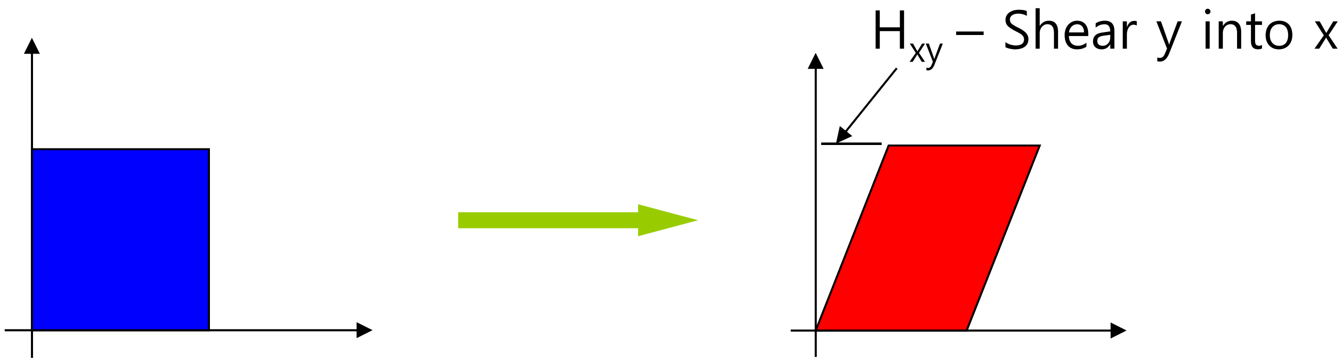
2D Shearing

- The Y-axis is not changed, and shearing applied in the X-axis direction:

  $x' = x + y \cdot h_{xy}$

  $y' = y$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & h_{xy} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



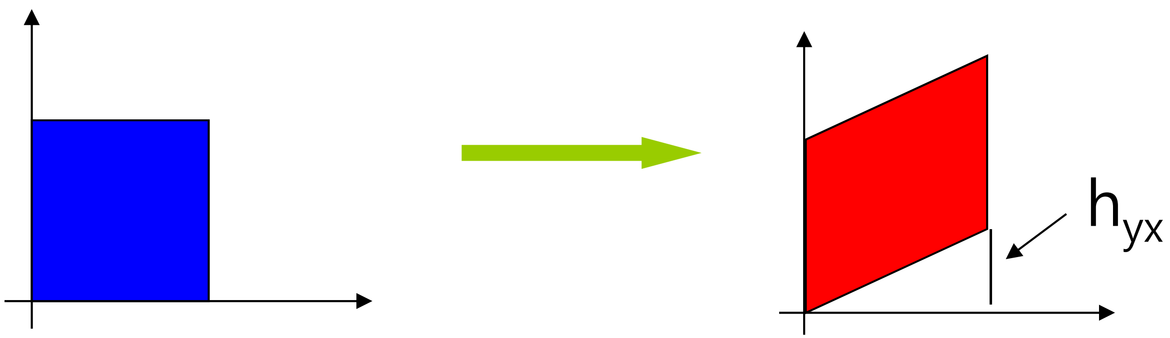
2D Shearing

- Shearing transformation does not change the size of object.
- The X-axis is not changed, and shearing applied in the Y-axis direction :

$x' = x$

$y' = x \cdot h_{yx} + y$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ h_{yx} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



Homogeneous Coordinates

- In order to multiply translation, rotation, scaling transformation matrix, homogeneous coordinates are used.
- In homogeneous coordinates, the two-dimensional point P (x, y) is expressed as P(x, y, w).
- (1, 2, 3) and (2, 4, 6) represent the same homogeneous coordinates.
- If the w of the point P (x, y, w) is 0, the point is located at an infinite point. If w is not 0, the point can be expressed as (x/w, y/w, 1).

Transforming Homogeneous Coordinates

$$T(dx, dy) = \begin{pmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{pmatrix}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S(sx, sy) = \begin{pmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The two-dimensional transformation matrix can be expressed as a 3x3 matrix of homogeneous coordinates.

3x3 2D Translation Matrix

- Matrix-vector multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

↓

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{pmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3x3 2D Rotation Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3x3 2D Scale Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3x3 2D Shearing Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} 1 & h_{xy} \\ h_{yx} & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

⬇

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} 1 & h_{xy} & 0 \\ h_{yx} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Inverse 2D Transformation Matrix

$$T^{-1} = \begin{pmatrix} 1 & 0 & -d_x \\ 0 & 1 & -d_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Composing Transformation

- *Composing transformation* is a process of forming one transformation by applying several transformation in sequence.
- If you want to transform one point, apply one transformation at a time or multiply the matrix and then multiply this matrix by the point.

$$Q = (M3 \cdot (M2 \cdot (M1 \cdot P))) = \underbrace{M3 \cdot M2 \cdot M1}_{M} \cdot P$$

(pre-multiply)

- Matrix multiplication is associative.

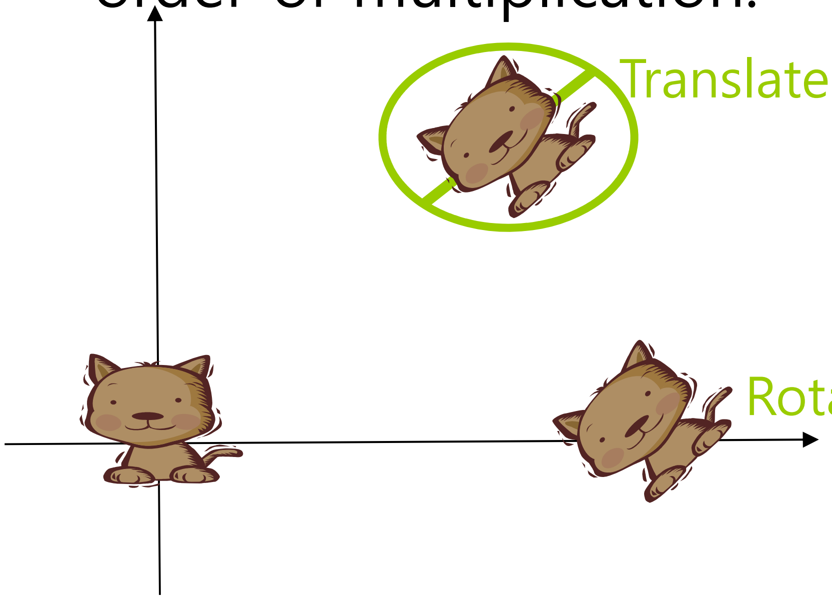
$$M3 \cdot M2 \cdot M1 = (M3 \cdot M2) \cdot M1 = M3 \cdot (M2 \cdot M1)$$

- Matrix multiplication is not commutative.

$$A \cdot B \ != B \cdot A$$

Transformation Order Matters!

- The multiplication of the transformation matrix is not commutative.
- Even if the transformation matrix is the same, it may have completely different results depending on the order of multiplication.



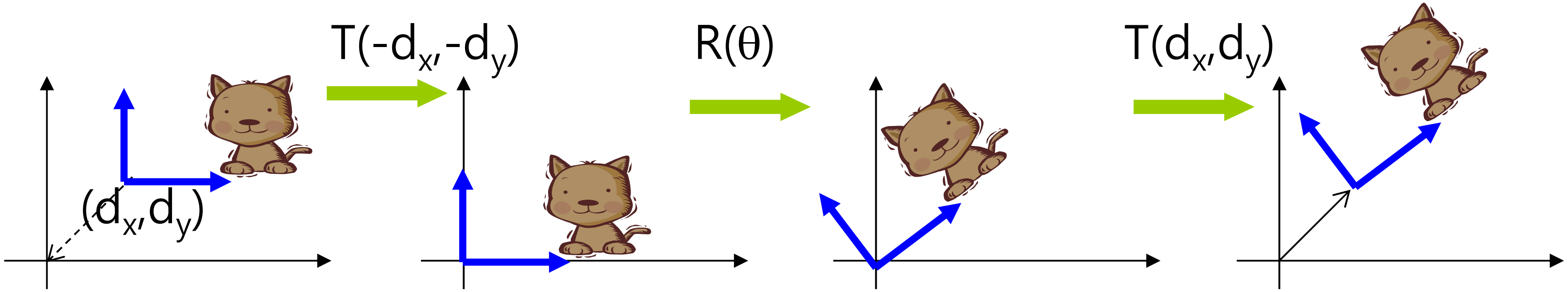
Translate (5, 0) and then rotate 60 degree

Rotate 60 degree and then translate (5,0)

2D Rotate about an Arbitrary Pivot

- Two-dimensional rotation by θ at an arbitrary pivot point $P(d_x, d_y)$:
  1. $T(-d_x, -d_y)$
  2. $R(\theta)$
  3. $T(d_x, d_y)$

$$\begin{pmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -d_x \\ 0 & 1 & -d_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & d_x(1-\cos\theta)+d_y\sin\theta \\ \sin\theta & \cos\theta & d_y(1-\cos\theta)-d_x\sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

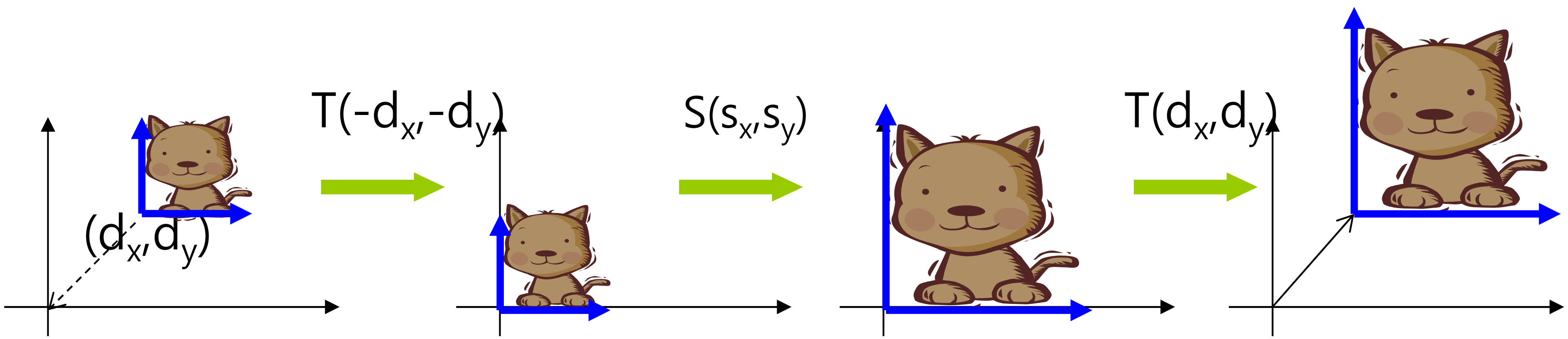


$T(-d_x,-d_y)\cdot R(\theta)\cdot T(d_x,d_y) = R(d_x, d_y, \theta)$

2D Scale about an Arbitrary Pivot

- Two-dimensional scaling an arbitrary pivot point $P(d_x, d_y)$:
  1. $T(-d_x, -d_y)$
  2. $S(s_x, s_y)$
  3. $T(d_x, d_y)$

$$\begin{pmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -d_x \\ 0 & 1 & -d_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & d_x(1-s_x) \\ 0 & s_y & d_y(1-s_y) \\ 0 & 0 & 1 \end{pmatrix}$$

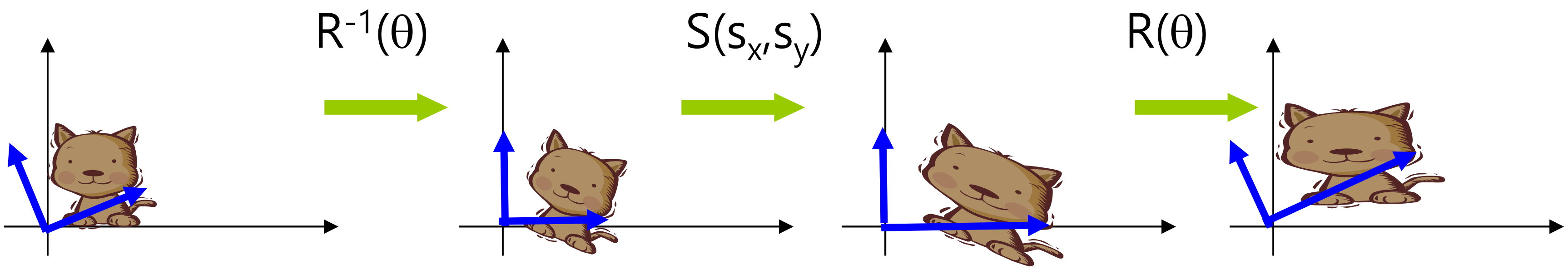


$T(-d_x,-d_y) \cdot S(s_x, s_y) \cdot T(d_x,d_y) = S(d_x, d_y, s_x, s_y)$

2D Scale in an Arbitrary Direction

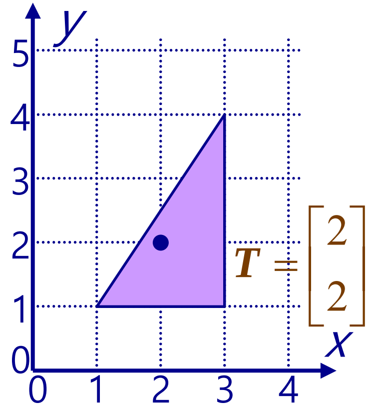
- Two dimensional scaling in an arbitrary direction (Rotating *the object to align the desired scaling directions with the coordinate axes* before scale transformation)
  1. $R^{-1}(\theta)$
  2. $S(s_x, s_y)$
  3. $R(\theta)$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_x\cos^2\theta + s_y\sin^2\theta & (s_x - s_y)\cos\theta\sin\theta & 0 \\ (s_x - s_y)\cos\theta\sin\theta & s_y\cos^2\theta + s_x\sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Example: 2D Rotate about an Arbitrary Pivot

Rotate a triangle with vertices (1,1), (3,1), (3,4) by 45 degrees about the pivot point (2,2).



1. Translate point to origin $\mathbf{T}_1 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$
2. Rotate 45 degrees $\mathbf{R} = \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix}$
3. Translate back to original location $\mathbf{T}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
4. Composite transformation $P' = R(P + T_1) + T_2$

$$P' = \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Example: 2D Rotate about an Arbitrary Pivot

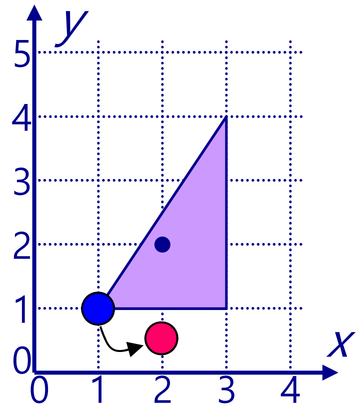
- P_1 (1, 1)

$$P_1' = \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1.414 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0.586 \end{bmatrix}$$



Example: 2D Rotate about an Arbitrary Pivot

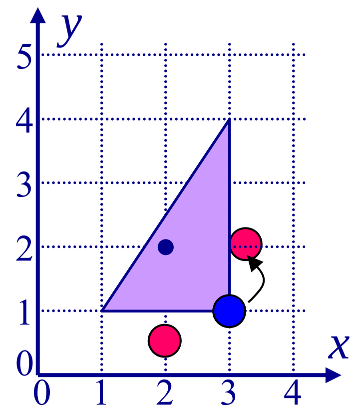
- P_2 (3, 1)

$$P_2' = \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.414 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3.414 \\ 2 \end{bmatrix}$$



Example: 2D Rotate about an Arbitrary Pivot

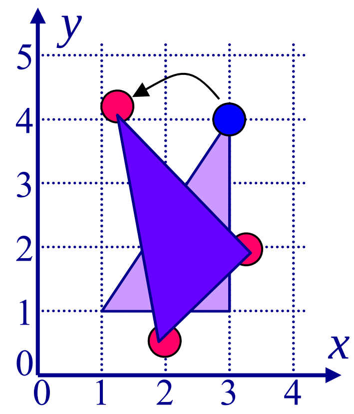
- P_3 (3, 4)

$$P_3' = \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

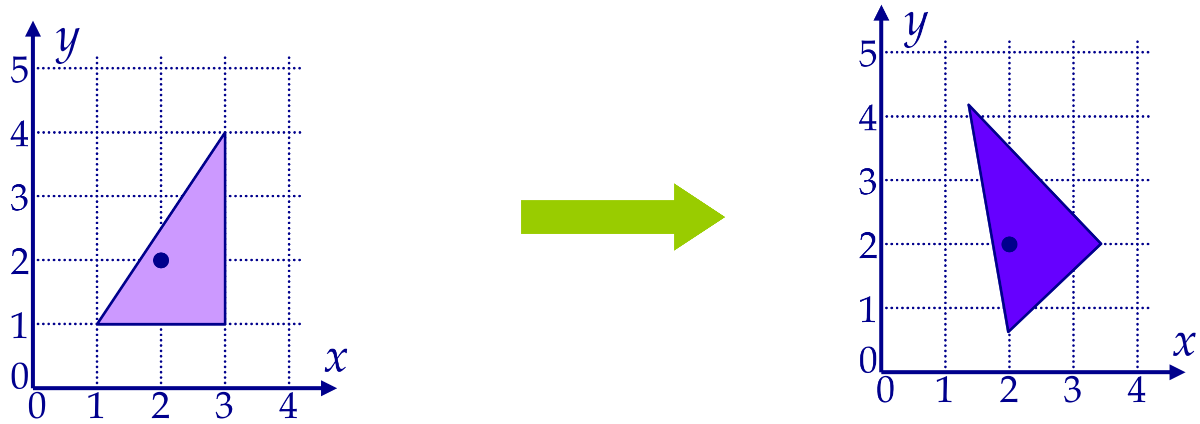
$$= \begin{bmatrix} -.707 \\ 2.121 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.293 \\ 4.121 \end{bmatrix}$$



Example: 2D Rotate about an Arbitrary Pivot

- Result:



Before:
(1, 1), (3, 1), (3, 4)

After:
(2, 0.59), (3.41, 2), (1.29, 4.2)

Example: 2D Rotate about an Arbitrary Pivot Using Composite Transformation Matrix

- Rotate a triangle with vertices (1,1), (3,1), (3,4) by 45 degrees about the pivot point (2,2).
- P′= T(2,2)R(45)T(-2,-2)P = M P

$$M = T_{(2,2)} R_{45} T_{(-2,-2)})$$

$$= \left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(45°) & -\sin(45°) & 0 \\ \sin(45°) & \cos(45°) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & -.828 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{M}$$

Example: 2D Rotate about an Arbitrary Pivot Using Composite Transformation Matrix

1. P_1

$$P_1' = MP_1 = \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & -.828 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ .586 \\ 1 \end{bmatrix}$$

2. P_2

$$P_2' = MP_2 = \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & -.828 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.414 \\ 2 \\ 1 \end{bmatrix}$$

3. P_3

$$P_3' = MP_3 = \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & -.828 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.293 \\ 4.121 \\ 1 \end{bmatrix}$$